ON THE FORMALISM OF SHIMURA VARIETIES

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To Dick Gross with admiration and gratitude

1. Introduction

The formalism of Shimura varieties was laid out by Deligne [D1], [D2] and Langlands [L] 45 years ago. The formalism they suggested seems to us to have a number of possible shortcomings:

(1) Deligne's 'Shimura datum', a pair (G, X) of a connected reductive group over $\mathbb{Q}$ and a $G(\mathbb{R})$ conjugacy class of homormorphisms $h : \mathrm{RS}^{\mathbb{R}}_{\mathbb{C}}\mathbb{G}_m \to G$ over $\mathbb{R}$ satisfying certain axioms, paremtrizes not a (inverse system of) varieties $\mathrm{Sh}(G, X)$ over some number field $E(G, X)$, but the pair $(\mathrm{Sh}(G, X)/E(G, X), \rho^{\mathrm{can}} : E(G, X) \hookrightarrow \mathbb{C})$ of the Shimura variety together with an embedding of its field of definition into $\mathbb{C}$. Indeed the 'same (inverse system of) varieties' over E can be parametrized by different Shimura data depending on the choice of embedding $E \hookrightarrow \mathbb{C}$.

(2) The theory of conjugation of Shimura varieties conjectured by Langlands [L] and established by Milne [Mi1] depends for its formulation on some unmotivated, and somewhat non-canonical, choices of cocycles, which to the best of our knowledge are written down only in [L]. This makes it quite hard to work with, as does its reliance of choices of special points.

(3) In [D2], Deligne imposes an axiom that the group G^{ad} should have no simple factor over $\mathbb{Q}$, whose real points are compact. This allows him to use strong approximation to explicitly understand the connected components of his Shimura varieties, but it should be unnecessary for their existence and for the study of their conjugation properties.

The third of these points is unrelated to the other two and will be easily remedied in section 3.4. We will discuss it no further in this introduction.

As a simple illustration point (1), consider a non-Galois totally real cubic extension $F/\mathbb{Q}$. It has three different embeddings $\tau_i : F \hookrightarrow \mathbb{R}$ for $i = 1, 2, 3$. Write ∞_i for the infinite place of F corresponding to τ_i. Let D_i/F denote the quaternion algebra

J.S. was supported partially by the National Science Foundation Graduate Research Fellowship (under Grant No. DGE-1656466), and partially by funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 884596).

R.T. was supported in part by NSF grant DMS-1902265.

centre F ramified at exactly ∞_j for $j \neq i$. Denote by $G_i/\mathbb{Q}$ the reductive groups with $G_i(\mathbb{Q}) = D_i^\times$. These groups are not isomorphic over $\mathbb{Q}$. We have $G_i(\mathbb{R}) \cong GL_2(\mathbb{R}) \times \mathbb{H}^\times \times \mathbb{H}^\times$, where $\mathbb{H}$ denotes the Hamiltonian quaternions. Let X_i denote the $G_i(\mathbb{R})$-conjugacy class of the morphism $h_i : \mathrm{RS}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_i$ defined over $\mathbb{R}$ with

$$h_i(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \times 1 \times 1.$$

We have $E(G_i, X_i) = \tau_i F \subset \mathbb{C}$. Note that $G_i \times \mathbb{A}^\infty$ is independent of i. We will denote the group $G_i(\mathbb{A}^\infty)$, which does not depend on i, simply as Γ. Deligne's theory of Shimura varieties gives us for each i an inverse system $\{\mathrm{Sh}(G_i, X_i)_U\}$ of varieties over $\tau_i F \subset \mathbb{C}$ indexed by sufficiently small open compact subgroups of Γ and with an action of Γ. However there is one such system $\{S_U\}$ of varieties over F indexed by sufficiently small open compact subgroups of Γ and with an action of Γ, such that $\{\tau_i S_U\}$ with its Γ-action is identified with $\{\mathrm{Sh}(G_i, X_i)_U\}$ with its Γ-action. It seems to us unnecessarily cumbersome and confusing to index the one system $\{S_U\}$ over F by three different Shimura data, depending on how one wants to view F as a subfield of $\mathbb{C}$. It would seem to be preferable to index $\{S_U\}$ by some other data $\mathcal{D}$ over F and then to give a recipe that to $\mathcal{D}$ and any embedding $\tau : F \hookrightarrow \mathbb{C}$ attaches a Deligne Shimura datum $(G_{\mathcal{D},\tau}, X_{\mathcal{D},\tau})$ so that

$$(\tau S_U)(\mathbb{C}) = G_{\mathcal{D},\tau} \backslash (G_{\mathcal{D},\tau}(\mathbb{A}^\infty)/U \times X_{\mathcal{D},\tau}).$$

It turns out that points (1) and (2) above are closely related. Indeed the second only became apparent to us as we tried to understand the first, and once we felt we understood the second, the first was easily remedied.

To us the key to understanding possible shortcomings (1) and (2) is, perhaps not surprisingly, to make use of Kottiwitz's cohomology groups $B(G)$. However it will be essential for us to work with 1-cocycles, not only 1-cohomology classes. We work out the requisite theory of cocycles in [ST].

In the rest of this introduction we will first recall Kottwitz's theory including a discussion of cocycles. We will then explain our, hopefully more canonical, reformulation of the theory of conjugation of Deligne's Shimura varieties. Finally we will state an alternative formulation which avoids the shortcoming (1).

1.1. **Algebraic cohomology.** If $G/\mathbb{Q}$ is an algebraic group then Kottwitz defines $B(\mathbb{Q}, G)_{\mathrm{basic}}$ as a direct limit over finite Galois extensions $E/\mathbb{Q}$ of algebraic cohomology pointed sets $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$, which contain $H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$. These pointed sets were canonically defined by Kottwitz [K2], but to define underlying sets of cocyles we need additional data. This was explained in [ST]. To explain our main results, we must recall some of this theory.

If G is an algebraic group we will write $Z(G)$ for its centre and $G^{\mathrm{ad}} = G/Z(G)$. If G is reductive we will write Λ_G for its arithmetic fundamental group.

If E is a number field, we will write V_E for the set of places of E, $\mathbb{Z}[V_E]$ for the free abelian group on V_E and $\mathbb{Z}[V_E]_0$ for the subgroup consisting of sums $\sum_w m_w w$ where $\sum_w m_w = 0$. If E is Galois over $\mathbb{Q}$, these all have natural $\mathrm{Gal}\,(E/\mathbb{Q})$-actions. In this case, we will write $T_{2,E}$ (resp. $T_{3,E}$) for the pro-torus over $\mathbb{Q}$ with character group $\mathbb{Z}[V_E]$ (resp. $\mathbb{Z}[V_E]_0$). There is a natural short exact sequence

$$(0) \longrightarrow \mathbb{G}_m \longrightarrow T_{2,E} \longrightarrow T_{3,E} \longrightarrow (0).$$

There is a map

$$\boldsymbol{\nu} : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} \longrightarrow \mathrm{Hom}\,(T_{3,E}, Z(G))(\mathbb{Q})$$

with kernel $H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$.

If G is reductive and split by E, then Kottwitz also defined an important map, the 'Kottwitz map',

$$\kappa : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} \longrightarrow (\mathbb{Z}[V_E]_0 \otimes \Lambda_G)_{\mathrm{Gal}\,(E/\mathbb{Q})},$$

where Λ_G denotes the arithmetic fundamental group of G.

The extra data we require is a choice of an element $\mathfrak{a}^+$ from a certain set $\mathcal{H}(E/\mathbb{Q})^+$, which has a transitive action of $T_{2,E}(\mathbb{A}_E)$. If S is a set of places of $\mathbb{Q}$ and if $G/\mathbb{Q}$ is an algebraic group, we obtain:

(1) Pointed sets $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}}$ with an action of $G(E)$ and $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S_{\mathfrak{a}^+}, G(\mathbb{A}_E^S))_{\mathrm{basic}}$ with an action of $G(\mathbb{A}_E^S)$ together with a $G(E)$-equivariant map

$$\mathrm{loc}_{\mathfrak{a}^+} : Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S_{\mathfrak{a}^+}, G(\mathbb{A}_E))_{\mathrm{basic}}.$$

These constructions are functorial in G. If $S = \emptyset$ we drop it from the notation.

(2) We have

$$H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} = G(E)\backslash Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}},$$

and the preimage of $H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$ is $Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$. Thus we have a map

$$Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}} \longrightarrow Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), G^{\mathrm{ad}}(E)),$$

which is surjective if $Z(G)$ is a torus. If $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}}$ then ϕ maps to an element of $Z^1(\mathrm{Gal}\,(E/F), G^{\mathrm{ad}}(E))$ and hence we obtain an inner form ${}^{\phi}G$ of G.

Similarly

$$H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\mathrm{basic}} = G(\mathbb{A}_E^S)\backslash Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S_{\mathfrak{a}^+}, G(\mathbb{A}_E^S))_{\mathrm{basic}}$$

is canonically independent of $\mathfrak{a}^+$; and there is a map

$$\boldsymbol{\nu} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\mathrm{basic}} \longrightarrow \bigoplus_{v \notin S} X_*(Z(G))(\mathbb{Q}_v),$$

with kernel (in $Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\rm basic}$) identified with $Z^1({\rm Gal}\,(E/\mathbb{Q}), G(\mathbb{A}_E^S))$. If $\phi \in Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\rm basic}$ we obtain an inner for ${}^\phi G$ of G over $\mathbb{A}_E^S$.

(3) If $S' \supset S$ there is a natural map

$$\mathrm{res}^{S'} : Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^S_{\mathfrak{a}^+}, G(\mathbb{A}_{E,S}))_{\rm basic} \longrightarrow Z^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^{S'}_{\mathfrak{a}^+}, G(\mathbb{A}_E^{S'}))_{\rm basic}.$$

Suppose that E_v^0 is a finite extension of $\mathbb{Q}_v$ isomorphic to E_w for some, and hence any, $w|v$. Then if $v \notin S$ there is a map

$$\mathrm{res}_{E_v^0/\mathbb{Q}_v} : H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\rm basic} \longrightarrow H^1_{\rm alg}(\mathcal{E}(E_v^0/\mathbb{Q}_v), G(E_v^0))_{\rm basic},$$

where the latter set is Kottwitz's local algebraic cohomology pointed set. This gives an isomorphism

$$H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\rm basic} \cong {\prod_{v\notin S}}' H^1_{\rm alg}(\mathcal{E}(E_v^0/\mathbb{Q}_v), G(E_v^0))_{\rm basic}.$$

(Here the product is restricted with respect to the $H^1({\rm Gal}\,(E_v^0/\mathbb{Q}_v), G(\mathcal{O}_{E_v^0}))$.)

(4) If G is reductive and split by E, then there are 'Kottwitz maps'

$$\kappa : H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\rm basic} \longrightarrow (\mathbb{Z}[V_E] \otimes \Lambda_G)_{{\rm Gal}\,(E/\mathbb{Q})} = \bigoplus_v \Lambda_{G,{\rm Gal}\,(E_v^0/\mathbb{Q}_v)},$$

and

$$\overline{\kappa} : H^1_{\rm alg}(\mathcal{E}^{\rm loc}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\rm basic} \longrightarrow \Lambda_{G,{\rm Gal}\,(E/\mathbb{Q})}.$$

The former is induced by the local Kottwitz maps and is compatible with the global Kottwitz map and loc. The latter is the composition of the former with the map $(\mathbb{Z}[V_E] \otimes \Lambda_G)_{{\rm Gal}\,(E/\mathbb{Q})} \to \Lambda_{G,{\rm Gal}\,(E/\mathbb{Q})}$ induced by the sum of the coefficients map $\mathbb{Z}[V_E] \to \mathbb{Z}$. Thus $\overline{\kappa} \circ {\rm loc} = 0$.

(5) If E is imaginary and $T/\mathbb{Q}$ is a torus split by E and $\mu \in X_*(T)(\mathbb{C})$ and $\tau \in {\rm Aut}\,(\mathbb{C})$, the group of field theoretic automorphisms of $\mathbb{C}$, then there is a special element

$$\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \in T(\mathbb{A}_E^\infty)/T(E)\overline{T(\mathbb{Q})}.$$

As E and $\mathfrak{a}^+$ vary there is an explicit way of comparing these constructions. If $D \supset E$, $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$ and $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$, then the comparison of the constructions for $(E, \mathfrak{a}_E^+)$ and $(D, \mathfrak{a}_D^+)$ depends on the choice of an element $t \in T_{2,E}(\mathbb{A}_D)$.

This material, along with some other background, is summarized in section 2 of [ST].

1.2. **Some algebraic cohomology classes.** If $G/\mathbb{R}$ is a connected reductive group and $\mu : \mathbb{G}_m \to G$ over $\mathbb{C}$ we call μ *basic* if $\mu^c\mu$ factors through $Z(G)$; and *compactifying* if it is basic and in addition ${\rm ad}\,\mu(-1)$, which lies in $G^{\rm ad}(\mathbb{R})$, is a Cartan involution. If Y is a $G(\mathbb{R})$-conjugacy class of basic cocharacters then we obtain a class $\widehat{\boldsymbol{\lambda}}_G(Y) \in H^1_{\rm alg}(\mathcal{E}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$. (The class of the cocycle

$$\widehat{\lambda}_G(\mu) : \langle \mathbb{C}^\times,\, j :\; j^2 = -1,\; jzj^{-1} = {}^c z\rangle \longrightarrow G(\mathbb{C})$$

which sends z to $(\mu^c\mu)(z)$ and j to $\mu(-1)$ for any $\mu \in Y$.) If $\phi \in \widehat{\boldsymbol{\lambda}}_G(Y)$ then ${}^\phi G$ comes equipped with a natural basic $G(\mathbb{R})$-conjugacy class of basic cocharacters $Y_{\phi G}$. (In the case $\phi = \widehat{\lambda}_G(\mu)$ we have $Y_{\phi G} = [\mu]_{({}^\phi G)(\mathbb{R})}$.)

If $G^{\mathrm{ad}}(\mathbb{R})$ is compact and if C is a G-conjugacy class of cocharacters, then C contains a unique basic $G(\mathbb{R})$-conjugacy class $Y(C)$.

Suppose now that Y is a $G(\mathbb{R})$-conjugacy class of compactifying cocharacters of G and $\phi \in \widehat{\boldsymbol{\lambda}}_G(Y)$ and $\tau \in \mathrm{Aut}\,(\mathbb{C})$. We will write $[Y]_{\phi G}$ for the ${}^\phi G$-conjugacy class containing $Y_{\phi G}$. We set

$$\widehat{\boldsymbol{\lambda}}_G(Y - {}^\tau[Y]_{\phi G}) = \widehat{\boldsymbol{\lambda}}_{\phi G}(Y({}^\tau[Y]_{\phi G})^{-1})[\phi] \in H^1_{\mathrm{alg}}(\mathcal{E}(\mathbb{C}/\mathbb{R}), G).$$

If $\psi \in \widehat{\boldsymbol{\lambda}}_G(Y - {}^\tau[Y]_{\phi G})$, then ${}^{\tau,\psi}Y = (Y({}^\tau[Y]_{\phi G})^{-1})^{-1}_{\psi G}$ is a well defined $({}^\psi G)(\mathbb{R})$-conjugacy class of compactifying cocharacters of ${}^\psi G$.

We now turn to the global case. Suppose that $G/\mathbb{Q}$ is a connected reductive group. If G is split by E, then Kottwitz showed that an element of $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$ is determined by its images under κ and $\mathrm{res}_{E^0_\infty/\mathbb{R}} \circ \mathrm{loc}$.

Suppose now that Y is a $G(\mathbb{R})$-conjugacy class of compactifying cocharacters of $G_{/\mathbb{C}}$. If $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and if $E/\mathbb{Q}$ is a sufficiently large finite Galois extension, then there is a unique

$$\boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$$

such that

- $\kappa(\boldsymbol{\phi}_{G,Y,\tau}) = {}^{\rho^{-1}}\lambda_G(Y) \otimes (w(\rho) - w(\tau\rho))$, where $\rho : E \hookrightarrow \mathbb{C}$ and $w(\rho)$ denotes the corresponding infinite place of E (this is independent of the choice of ρ) and where $\lambda_G(Y)$ denotes the image of any element of Y in Λ_G;
- and $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = \widehat{\boldsymbol{\lambda}}_G(Y - {}^\tau[Y]_G)$.

If $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$ then ${}^\phi G$ comes with a canonical $({}^\phi G)(\mathbb{R})$ conjugacy class of compactifying cocharacters ${}^{\tau,\phi}Y$.

This material is discussed in sections 2.4 and 2.5.

1.3. **Conjugation of Deligne's Shimura varieties.** One can define a Shimura datum (in the sense of Deligne) to be a pair (G, Y), where $G/\mathbb{Q}$ is a connected reductive group and Y is a compactifying $G(\mathbb{R})$-conjugacy class of *miniscule* cocharacters $\mu : \mathbb{G}_m \to G_{/\mathbb{C}}$. It is more common to consider instead of Y a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathrm{RS}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_{/\mathbb{R}}$ satisfying certain properties, but these two notions are easily seen to be equivalent. (To a μ as above we associate h_μ which is the descent from $\mathbb{C}$ to $\mathbb{R}$ of $(\mu, {}^c\mu)$.) Also note that Deligne assumes that G^{ad} has no simple factor over $\mathbb{Q}$ whose real points are compact. However, as we will see, everything (that we will be discussing) remains true without this assumption.

To the Shimura datum (G, Y) and a sufficiently small open compact subgroup $U \subset G(\mathbb{A}^\infty)$, Deligne associates a smooth quasi-projective variety $\mathrm{Sh}(G, Y)_U/\mathbb{C}$ (called a

Shimura variety) together with an identification of complex manifolds

$$G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) \xrightarrow{\sim} \mathrm{Sh}(G,Y)_U(\mathbb{C}).$$

The system of these Shimura varieties as U varies has an action of $G(\mathbb{A}^\infty)$ (by right translation). If $f:(G,Y)\to(G',Y')$ is a morphism of Shimura data (i.e. a morphism $f:G\to G'$ of algebraic groups over $\mathbb{Q}$ which carries Y to Y') then there is an induced maps of Shimura varieties. Deligne defines the reflex field $E(G,Y)\subset\mathbb{C}$ to be the number field which is the fixed field of all automorphsims of $\mathbb{C}$ which fix the $G(\mathbb{C})$-conjugacy class $[Y]_{G(\mathbb{C})}$ of cocharacters of G, which contains Y. He conjectured that $\mathrm{Sh}(G,Y)_U$ has a model over $E(G,Y)$ satisfying certain additional properties, which determine it uniquely. He proved this in many cases and Milne proved it in all cases. Langlands conjectured a rather complicated and apparently ad hoc formula depending on a number of choices for the conjugate of $\mathrm{Sh}(G,Y)_U$ by any automorphism of $\mathbb{C}$. This was also proved by Milne.

Fix a sufficiently large finite Galois extension $E/\mathbb{Q}$ and $\mathfrak{a}^+\in\mathcal{H}(E/\mathbb{Q})^+$. If (G,Y) is a Shimura datum and $\phi\in\boldsymbol{\phi}_{G,Y,\tau}$, then $({}^\phi G,{}^{\tau,\phi}Y)$ is another Shimura datum. If moreover $b\in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}^\infty\mathrm{loc}_{\mathfrak{a}^+}\phi={}^b1$, then we will define an isomorphism

$$\Phi_{\mathfrak{a}^+}(\tau,\phi,b): {}^\tau\mathrm{Sh}(G,Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^\phi G,{}^{\tau,\phi}Y)_{bUb^{-1}}.$$

These maps commute with the action of $G(\mathbb{A}^\infty)$ (using the identification $\mathrm{conj}_b: G(\mathbb{A}^\infty)\xrightarrow{\sim}{}^\phi G(\mathbb{A}^\infty)$) and with the action of morphisms $f:(G,Y)\to(G',Y')$ of Shimura data. One has a cocycle relation

$$\Phi_{\mathfrak{a}^+}(\tau_1\tau_2,\phi_1\phi_2,b_1b_2)=\Phi_{\mathfrak{a}^+}(\tau_1,\phi_1,b_1)\circ{}^{\tau_1}\Phi_{\mathfrak{a}^+}(\tau_2,\phi_2,b_2).$$

In the case where $(G,Y)=(T,\{\mu\})$ with T a torus there is an explicit formula for the $\Phi_{\mathfrak{a}^+}(\tau,\phi,b)$ involving the elements $\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}$. These properties together completely (over) characterize the maps $\Phi_{\mathfrak{a}^+}(\tau,\phi,b)$. We also explain how the maps $\Phi_{\mathfrak{a}^+}(\tau,\phi,b)$ depend on E and $\mathfrak{a}^+$. (See theorem 3.5 for all this.)

In particular the maps $\Phi_{\mathfrak{a}^+}(\tau,1,1)$ for $\tau\in\mathrm{Aut}(\mathbb{C})$ fixing $E(G,Y)$ provide descent data for $\mathrm{Sh}(G,Y)_U$ from $\mathbb{C}$ to $E(G,Y)$, which yields the canonical model of $\mathrm{Sh}(G,Y)_U$ over $E(G,Y)$.

The conjugation morphisms, whose existence was conjectured by Langlands and proved by Milne, are special cases of our maps $\Phi_{\mathfrak{a}^+}(\tau,\phi,b)$ in which ϕ and b factor through a suitable maximal torus in G and take a very particular form. Indeed our theorem follows easily from Milne's theorem, once we were able to discover the correct formulation (and unravel Langlands definitions).

This is all discussed in section 3.

1.4. **Rational Shimura varieties.** Finally we propose an alternative formalism, which we feel is better suited to keeping track of the rationality properties of Shimura varieties.

Fix a sufficiently large Galois extension $E/\mathbb{Q}$ and $\mathfrak{a}^+ \in \mathcal{H}^+(E/\mathbb{Q})$. The theory we describe is independent of these choices, in a way that is described precisely in the body of the paper.

By a *rational Shimura datum* over a field L of characteristic 0 we mean a 4-tuple (G, ψ, C, U), where

- $G/\mathbb{Q}$ is a connected reductive group;
- $\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}^+}, G(\mathbb{A}_E))_{\mathrm{basic}}$ such that ${}^{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi}G^{\mathrm{ad}}(\mathbb{R})$ is compact;
- C is a conjugacy class of miniscule cocharacters of G (considered as a variety) defined over L such that the image of C in $\Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})}$ (which is independent of how one compares the fields E and L) equals $\overline{\kappa}_G(\psi)$;
- and $U \subset ({}^{\psi}G)(\mathbb{A}^\infty)$ is an open compact subgroup.

The group G plays very little role except as a basis point to identify the class of extended pure inner forms with which we are working. One gets a completely equivalent theory if one replaces G by ${}^{\phi}G$ for $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}}$. In the case that $Z(G)$ is connected, we may assume without loss of generality that G is quasi-split.

To a rational Shimura datum (G, ψ, C, U) over L we associate a normal (smooth if U is sufficiently small) quasi-projective variety $\mathrm{Sh}(G, \psi, C, U)/L$. As U varies the system of varieties has an action of $({}^{\psi}G)(\mathbb{A}^\infty)$. (Note that ${}^{\psi}G/\mathbb{A}$ may well not arise from a group over $\mathbb{Q}$, it is often what one might call 'incoherent'.)

Crucially the action of Galois on Shimura varieties for rational Shimura data becomes completely transparent. If $\tau : L \to L'$ then

$$\{{}^{\tau}\mathrm{Sh}(G, \psi, C, U)\}_U = \{\mathrm{Sh}(G, \psi, {}^{\tau}C, U)\}_U$$

(with their ${}^{\psi}G(\mathbb{A}^\infty)$-actions).

These rational Shimura varieties are not exactly equal to canonical models of Deligne's Shimura varieties, rather they are finite unions of isomorphic copies of a single such canonical model. Thus they carry the same information. Indeed when one describes Shimura varieties as moduli spaces over rings of mixed characteristics it is these rational Shimura varieties that arise, as has long been observed. (See for example [K1] and [HT] and section 5 of this paper.) An additional benefit is that these rational Shimura varieties actually have an action of a larger group than $({}^{\psi}G)(\mathbb{A}^\infty)$, a group that transitively permutes the constituent Deligne Shimura varieties. More precisely let $\widetilde{G}_{E,\psi}(\mathbb{A})$ denote the abelian group

$$\{(\zeta, g) \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \times G(\mathbb{A}_E) : \ (\mathrm{loc}_{\mathfrak{a}}\zeta)^g\psi = \psi\}$$

with componentwise multiplication. There are embeddings

$$\begin{array}{ccc} {}^{\psi}G(\mathbb{A}) & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}) \\ g & \longmapsto & (1, g) \end{array}$$

and

$$\begin{array}{ccc} Z(G)(E) & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}) \\ \delta & \longmapsto & ((\sigma \mapsto \delta/{}^{\sigma}\delta), \delta^{-1}). \end{array}$$

We define

$$\widetilde{G}_{E,\psi}(\mathbb{A}^\infty) = \widetilde{G}_{E,\psi}(\mathbb{A})/\overline{Z(G)(\mathbb{Q})^\psi G(\mathbb{R})}.$$

(The notation is not meant to suggest that $\widetilde{G}(\mathbb{A})$ or $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ are the $\mathbb{A}$ or $\mathbb{A}^\infty$ points of any algebraic group.) Then we have an exact sequence

$$(0) \longrightarrow {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_{E,\psi}(\mathbb{A}^\infty) \longrightarrow \ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E))) \longrightarrow (0).$$

The action of ${}^\psi G(\mathbb{A}^\infty)$ on the system of the $\{\mathrm{Sh}(G, \psi, C, U)\}_U$ extends to an action of $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$, which permutes transitively the constituent Deligne Shimura varieties.

Shimura varieties for rational Shimura data are also functorial in the rational Shimura data in the following sense: By a *morphism*

$$(\phi, g, f) : (G_1, \psi_1, C_1, U_1) \to (G_2, \psi_2, C_2, U_2)$$

of rational Shimura data over L, we mean

- a cocycle $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G_2(E))_{\mathrm{basic}}$,
- an element $g \in G_2(\mathbb{A}_E)$,
- and a morphism $f : G_1 \to {}^\phi G_2$ defined over $\mathbb{Q}$,

such that $f \circ \psi_1 = {}^{g^{-1}}\psi_2 \mathrm{loc}_{\mathfrak{a}}\phi^{-1}$ and $f(C_1) \subset C_2$ and $(\mathrm{conj}_g \circ f)(U_1) \subset U_2$. Given such a morphism we obtain a morphism a morphism of varieties over E:

$$\mathrm{Sh}(\phi, g, f) : \mathrm{Sh}(G_1, \psi_1, C_1, U_1) \longrightarrow \mathrm{Sh}(G_2, \psi_2, C_2, U_2).$$

(The case $\phi \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)$ and $f = 1$ recovers the action of $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$.) We have

$$\mathrm{Sh}(\phi_1, g_1, f_1) \circ \mathrm{Sh}(\phi_1, g_2, f_2) = \mathrm{Sh}(f_1(\phi_2)\phi_1, g_1 f_1(g_2), f_1 \circ f_2).$$

If $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}}$, then $\mathrm{Sh}(\phi, 1, 1)$ gives a canonical isomorphism between the the tower $\{\mathrm{Sh}(G, \psi, C, U)\}_U$ with its $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$-action and the alternative tower $\{\mathrm{Sh}({}^\phi G, \psi(\mathrm{loc}_{\mathfrak{a}^+}\phi)^{-1}, C, U)\}_U$ with its $\widetilde{{}^\phi G}_{E,\psi(\mathrm{loc}_{\mathfrak{a}}\phi)^{-1}}(\mathbb{A}^\infty) = \widetilde{G}_{G,\psi}(\mathbb{A}^\infty)$-action. Thus, as we have already mentioned, the exact choice of G amongst its class of inner forms is not so important.

For any $g \in G(\mathbb{A}_E)$ the map $\mathrm{Sh}(1, g, 1)$ gives an isomorphism between the system $\{\mathrm{Sh}(G, \psi, C, U)\}_U$ with its $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$-action and $\{\mathrm{Sh}(G, {}^g\psi, C, V)\}_V$ with its $\widetilde{G}_{E,{}^g\psi}(\mathbb{A}^\infty)$-action, where we use conjugation by g to identify $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ and $\widetilde{G}_{E,{}^g\psi}(\mathbb{A}^\infty)$. Thus in a sense $\{\mathrm{Sh}(G, \psi, C, U)\}_U$ only depends on $[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$. However the identification is not canonical - it depends on the choice of g taking ψ to ${}^g\psi$. This is why we have to work with cocycles and not only cohomology classes.

There is of course a theory of complex uniformization for rational Shimura varieties. If (G, ψ, C, U) is a rational Shimura datum over $\mathbb{C}$, then $\mathrm{Sh}(G, \psi, C, U)(\mathbb{C})$ admits a uniformization by an Hermitian symmetric space, but this depends on auxiliary choices. We must choose $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}^+}, G(E))_{\mathrm{basic}}$ and $b \in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}[\phi] = \widehat{\boldsymbol{\lambda}}_{\psi G}(Y(C)^{-1})\mathrm{res}_{\mathbb{C}/\mathbb{R}}[\psi]$ and $\mathrm{res}^\infty\mathrm{loc}_{\mathfrak{a}^+}\phi = {}^b\mathrm{res}^\infty\psi$. We will write ${}^\phi G(E)^{\mathbb{Q}}_{\mathbb{R}}$ for the subgroup of elements of ${}^\phi G(E)$ whose image in ${}^\phi G^{\mathrm{ad}}(E)$ lies in

${}^{\phi}G^{\rm ad}(\mathbb{Q})$ and can be lifted to ${}^{\phi}G(\mathbb{R})$. Then ${}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}$ acts on $Y({}^{\tau}C)$ and there is a mapping ${}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}} \to \widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$. There is an isomorphism of complex manifolds

$$\pi_{(\phi,b)} : {}^{\phi}G^{\rm ad}(\mathbb{Q})_{E,\mathbb{R}} \backslash (\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})/U \times Y({}^{\tau}C)) \xrightarrow{\sim} {}^{\rho}{\rm Sh}(G,\psi,C)_U(\mathbb{C}).$$

In the case that $G = T$ is a torus the action of ${\rm Aut}\,(\mathbb{C})$ can be made explicit: if $\tau \in {\rm Aut}\,(\mathbb{C})$ then

$$\tau(\pi_{(\phi,b)}(\widetilde{g},\mu)) = \pi_{(\phi_{\tau}\phi,b_{\tau}b)}(\widetilde{g},{}^{\tau}\mu),$$

for any $\phi_{\tau} \in \boldsymbol{\phi}_{T,\{\mu\},\tau}$ and $b_{\tau} \in T(\mathbb{A}_E^{\infty})/\overline{T(\mathbb{Q})}$ such that ${\rm res}^{\infty}{\rm loc}_{\mathfrak{a}}\phi_{\tau} = {}^{b_{\tau}}1$ and the image of b_{τ} in $T(\mathbb{A}_E^{\infty})/\overline{T(\mathbb{Q})}T(E)$ is $\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}$.

We will write $Z(G)^1$ for the torus which is the intersection over all open subgroups $U \subset Z(G)(\mathbb{A}^{\infty})$ of the Zariski closure of $Z(G)(\mathbb{Q}) \cap U$. (If the $\mathbb{Q}$ split rank of $Z(G)$ equals the $\mathbb{R}$ split rank, then $Z(G)^1$ is trivial. See theorem 5.12 of [PR].) If (G,ψ,C,U) is a rational Shimura dataum over $\mathbb{C}$ and if $\boldsymbol{r}$ is a representation of ${}^{\psi}G/Z(G)^1$ on a finite free $\mathbb{A}^{\infty}$ module $\boldsymbol{W}$ which is *rationalizable* in a sense defined in section 4.6, then there is a canonical variation of rational Hodge structures $\mathcal{W}_{\boldsymbol{r}}/{\rm Sh}(G,\psi,C,U)(\mathbb{C})$, with $\mathcal{W}_{\boldsymbol{r}} \otimes_{\mathbb{Q}} \mathbb{R}$ polarizable. This variation of Hodge structures is unique, but only up to an isomorphism that is unique only up to composition with an element of $Z(\mathbb{Q})$, for some linear algebraic group $Z/\mathbb{Q}$ with $Z(\mathbb{A}^{\infty})$ equal to the centralizer of $\boldsymbol{r}$ in $GL(\boldsymbol{W})$. (So if $\boldsymbol{r}$ is absolutely irreducible, up to scalar multiples.)

All this is discussed in section 4. For a complete statement of the results mentioned here see theorem 4.3 and section 4.6.

At the suggestion of Pol van Hoften we included section 5.1, where we show that the PEL moduli spaces of type A and C considered by Kottwitz in [K1] are rational Shimura varieties in our sense.

1.5. **Acknowledgements.** After we released the first version of this paper Xinwen Zhu informed us that he and Liang Xiao had some similar results. In particular they had also removed the condition in Milne's theorem that $G^{\rm ad}$ has no $\mathbb{Q}$-simple factor whose real points are compact. They had also considered finite unions of Deligne's Shimura varieties with an action of a group similar to (and perhaps equal to) $\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$ and found these to have, in some ways, better properties.

We were aware of Dick Gross' paper [G1], and Dick has since made us aware of the follow up paper [G2], where related ideas are discussed in special cases. It is a great pleasure to dedicate this paper to him.

2. Algebraic background

2.1. **Notations.** *For simplicity we will assume all fields we consider in this paper will be assumed to be perfect unless we specifically say otherwise.*

If V is a finite dimensional vector space we will write $V^\vee$ for its dual. If $(\ ,\ )$ is a perfect pairing on V we will write $(\ ,\ )^\vee$ for the perfect pairing on $V^\vee$ characterized by

$$((x,\ ),(y,\ ))^\vee = (y,x).$$

Then $(\ ,\ )^{\vee\vee} = (\ ,\ )$ under the canonical identification $V \xrightarrow{\sim} V^{\vee\vee}$.

If F is a field we will write $\overline{F}$ for an algebraic closure of F and F^{ab} for the maximal abelian Galois extension of F in $\overline{F}$. If E/F is a Galois extension and L/F any field extension, then we will write $E \cap L$ (resp. EL) for $\rho(E) \cap L$ (resp. $\rho(E)L$) for any F-linear embedding $\rho : E \hookrightarrow \overline{L}$. The field $E \cap L$ (resp. EL) is a subfield of L (resp. $\overline{L}$) independent of the choice of ρ, but the identification of $E \cap L$ with a subfield of E depends on ρ. If L is any field of characteristic 0, we will write L^{alg} for the subfield consisting of elements algebraic over $\mathbb{Q}$. If L/K is any extension of fields we will write $\mathrm{Aut}\,(L/K)$ for the group of field theoretic automorphisms of L which fix K pointwise. If L has characteristic 0 (resp. $p > 0$) will write $\mathrm{Aut}\,(L)$ for $\mathrm{Aut}\,(L/\mathbb{Q})$ (resp. $\mathrm{Aut}\,(L/\mathbb{F}_p)$). If L/K is Galois we will write $\mathrm{Gal}\,(L/K)$ for $\mathrm{Aut}\,(L/K)$. If E is a subfield of $\mathbb{C}^{\mathrm{alg}}$ then

$$\mathbb{C}^{\mathrm{Aut}\,(\mathbb{C}/E)} = E.$$

If F is a local field of characteristic 0 we will write $\mathrm{Art}_F : F^\times \to \mathrm{Gal}\,(F^{\mathrm{ab}}/F)$ for the Artin map. (Normalized to take uniformizers to geometric Frobenius elements.)

If F is an algebraic extension of $\mathbb{Q}$ we will write V_F for the set of places of F and $\mathbb{A}_F$ for the ring of adeles of F. (In the case that F is an infinite extension of $\mathbb{Q}$ then $\mathbb{A}_F = \lim_{\to E} \mathbb{A}_E$, where E runs over subfields of F finite over $\mathbb{Q}$.) If v is a place of F then by F_v we will mean $\lim_{\to E} E_v$ as E runs over subextensions of $F/\mathbb{Q}$ which are finite over $\mathbb{Q}$. (So F_v may not be complete, but it is algebraic over $\mathbb{Q}_v$.) If F is a number field will write $\mathrm{Art}_F : \mathbb{A}_F^\times/\overline{F^\times(F_\infty^\times)^0} \xrightarrow{\sim} \mathrm{Gal}\,(F^{\mathrm{ab}}/F)$ for the Artin map.

If E/F is an algebraic extension of fields with F a number field and if $S \subset V_F$ we will write $V_{E,S}$ for the set of places of E above a place in S, and $\mathbb{A}_{E,S}$ for the ring of adeles of E supported at the primes in S. (If E is also a number field then $\mathbb{A}_{E,S}$ is the restricted product $\prod'_{w:\ w|_F \in S} E_w^\times$.) We will also write $V_E^{V_F - S} = V_{E,S}$ and $\mathbb{A}_E^{V_F-S} = \mathbb{A}_{E,S}$.

We will write $\mathbb{Z}[V_{E,S}]$ for the free ablelian group on $V_{E,S}$ and $\mathbb{Z}[V_{E,S}]_0$ for the subabelian group consisting of elements $\sum_w m_w w$ with $\sum_w m_w = 0$. If E/F is Galois, both groups have a natural action of $\mathrm{Gal}\,(E/F)$ via $\sigma \sum_w m_w w = \sum_w m_w(\sigma w) = \sum_w m_{\sigma^{-1}w} w$.

If F is an algebraic extension of $\mathbb{Q}$ and K is a local field and $\rho : F \hookrightarrow \overline{K}$, then we will write $v(\rho) = v(F,\rho)$ or $w(\rho) = w(F,\rho)$ or $u(\rho) = u(F,\rho)$ for the place of F induced by ρ. (We will tend to use $v(\rho)$ when the field is denoted F, $w(\rho)$ when it is denoted E and $u(\rho)$ otherwise.) If moreover $F/\mathbb{Q}$ is Galois and $\tau \in \mathrm{Aut}\,(\overline{K})$, then we will write τ^ρ for the element of $\mathrm{Gal}\,(F/\mathbb{Q})$ satisfying $\rho \circ \tau^\rho = \tau\rho$.

If E/F is a Galois extension with F a number field, and if v is a real place of F we will write $[c_v]$ for the conjugacy class in $\mathrm{Gal}\,(E/F)$ consisting of complex conjugations at places above v. If $F = \mathbb{Q}$ and $v = \infty$ we will simply write $[c]$.

2.2. **Algebraic groups.** If G is an (algebraic) group then $Z(G)$ will denote its centre and G^{ad} will denote $G/Z(G)$. Moreover G^{der} will denote its commutator subgroup and $C(G) = G^{\mathrm{ab}}$ will denote it co-center/abelianization G/G^{der}. If $H \subset G$ is a subgroup we will write $N_G(H)$ for its normalizer and $Z_G(H)$ its centralizer. If H has finite index, we will also write $\mathrm{tr}_{\,G/H} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ for the transfer map. If H is normal in G, then the image of $\mathrm{tr}_{\,G/H}$ is contained in $(H^{\mathrm{ab}})^{G/H}$. If G acts on X we will write $[x]_G$ for the G orbit of $x \in X$ and $Z_G(x)$ or G_x or $\mathrm{Stab}_G(x)$ for the stabilizer of x in G. If G is an algebraic group acting on a variety X over a field F and $x \in X(F)$, then $[x]_G$ is a variety, and $[x]_G(F) \supset [x]_{G(F)}$, but these two sets may not be equal.

If F is a field, if $K_1,\dots,K_r$ are fields containing F, and if G/F is an algebraic group; then we will write $G^{\mathrm{ad}}\,(F)_{K_1,\dots,K_r}$ for the subgroup of $G(F)$ consisting of elements which admit lifts to each $G(K_i)$. If E/F is a finite extension of fields we will write $G(E)^F_{K_1,\dots,K_r}$ for the set of elements of $G(E)$ that map to $G^{\mathrm{ad}}\,(F)_{K_1,\dots,K_r} \subset G^{\mathrm{ad}}\,(E)$.

We will write std for the character $t \mapsto t$ of $\mathbb{G}_m$.

If G is an affine algebraic group over F then there a scheme $X_*(G)$, smooth and separated over F, and a homomorphism $\mu^{\mathrm{univ}} : \mathbb{G}_m \times_F X_*(G) \to G \times_F X_*(G)$, such that if S is any F-scheme and $\mu : \mathbb{G}_{m,S} \to G_S$ is a homomorphism, then there is a unique morphism $S \to X_*(G)$ under which μ^{univ} pulls back to μ. Moreover

$$\begin{array}{ccc} G \times X_*(G) & \longrightarrow & X_*(G) \times X_*(G) \\ (g,\mu) & \longmapsto & (\mathrm{conj}_g \circ \mu, \mu) \end{array}$$

is smooth; and

$$X_*(G)_{\overline{F}} = \coprod_{[\mu] \in G(\overline{F}) \backslash X_*(G)(\overline{F})} G/Z_G(\mu).$$

(See sections 4 and 5 of exposé XI in [SGA3].) If G is geometrically connected, then the $G/Z_G(\mu)$ are the connected components of $X_*(G)$. Moreover if $\mu \in X_*(G)(\overline{F})$ and if $F([\mu])$ denotes the fixed field of $\mathrm{Stab}_{\mathrm{Gal}\,(\overline{F}/F)}([\mu]_{G(\overline{F})})$, then $X_*(G)_{F([\mu])}$ has a (unique) connected component $[\mu]$ such that $[\mu](\overline{F}) = [\mu]_{G(\overline{F})}$. (Use lemma 33.7.18 of [Stacks].) If $C \subset X_*(G)$ we will write C^{-1} for the set of μ^{-1}, where $\mu \in C$.

We will require all our reductive groups to be geometrically connected, i.e. by the term 'reductive group' we will mean what is often referred to as 'connected reductive group'. If G is reductive we will write G^{SC} for the simply connected semi-simple cover of G^{der}. If T is a maximal torus of G we will write T^{ad} for the image of T in G^{ad} (a maximal torus in G^{ad}) and $T^{\mathrm{der}} = (G^{\mathrm{der}} \cap T)$ (which is a maximal torus in G^{der}, see remark 3.5 of [Co]) and T^{SC} for the preimage of T in G^{SC} (which is a maximal torus in G^{SC}, see for instance proposition 4.1 of [Co]). We have $T = Z_G(T)$. We will also write W_T for the Weyl group $N_G(T)/T$, which we think of a finite algebraic group. It acts faithfully on T. We will also write $W_{T,F}$ for $N_G(T)(F)/T(F) \subset W_T(F)$.

If G is reductive and $T \subset G$ is a torus then the centralizer $Z_G(T)$ is (connected) reductive. (See corollary 11.12 of [B1].)

We remark that if $T \subset G$ is a maximal torus and $\mu_1, \mu_2 \in X_*(T)$ are conjugate under $G(\overline{F})$ then they are conjugate under $W_T(\overline{F})$. (This is probably well known, but as we don't know a reference we will sketch the proof. Let H denote the centralizer of $\mu_1(\mathbb{G}_m)$ in G. It is reductive. (See theorem 2.1 of [Co].) Suppose that $\mu_1 = g\mu_2 g^{-1}$. Then $\mu_1(\mathbb{G}_m) \subset gTg^{-1}$ so that T and gTg^{-1} are both maximal tori in H. Hence we have $gTg^{-1} = hTh^{-1}$ for some $h \in H$. Then $h^{-1}g \in N_G(T)$ and $\mu_1 = h^{-1}g\mu_2 g^{-1}h$, as desired.)

We will let Λ_G denote the algebraic fundamental group of G, i.e. $X_*(T)/X_*(T^{\mathrm{SC}})$ for any maximal torus T of G. Note that the Weyl group W_T acts trivially on $X_*(T)/X_*(T^{\mathrm{SC}})$. Any two maximal tori T and T' defined over F are conjugate over the separable closure $\overline{F}$ of F by $g \in G(\overline{F})$ with $gN_G(T)$ uniquely defined. Then conj_g induces an isomorphism $X_*(T)/X_*(T^{\mathrm{SC}}) \xrightarrow{\sim} X_*(T')/X_*(T'^{\mathrm{SC}})$. If we alter g by an element $h \in N_G(T)(\overline{F})$ then this isomorphism changes by an element of $W_T(\overline{F})$, i.e. is in fact unchanged. Thus Λ_G is canonically defined independent of the choice of T. In particular it has a canonical action of $\mathrm{Gal}\,(\overline{F}/F)$. (If $T' = \mathrm{conj}_g T$ and $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$, then $\sigma \circ \mathrm{conj}_g = \mathrm{conj}_g \circ \sigma \circ \mathrm{conj}_{w_\sigma}$ on $X_*(T)$ for some $w_\sigma \in W_T(\overline{F})$, and so $\sigma \circ \mathrm{conj}_g = \mathrm{conj}_g \circ \sigma$ on Λ_T.) If $[\mu]$ is a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G$, then $[\mu]$ gives rise to well defined element $\lambda_G([\mu]) \in \Lambda_G$. If $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$ then $\lambda_G({}^\sigma[\mu]) = {}^\sigma\lambda_G([\mu])$.

If C is a G-conjugacy class of cocharacters of G and $\phi \in Z^1(\mathrm{Gal}\,(E/F), G^{\mathrm{ad}}(E))$, then we set $C_{\phi G}$ to be the image of C under the identification $\iota_\phi : G \times E \xrightarrow{\sim} {}^\phi G \times E$, a conjugacy class of ${}^\phi G$. Under the identification $\mathrm{conj}_g : {}^\phi G \xrightarrow{\sim} {}^{g\phi}G$, the conjugacy class $C_{\phi G}$ is sent to $C_{{}^{g\phi}G}$. Moreover if $\sigma \in \mathrm{Gal}\,(\overline{F}/F)$, then we have ${}^\sigma(C_{\phi G}) = ({}^\sigma C)_{\phi G}$.

Now suppose that F is a number field and G/F is a connected algebraic group. Then $G(F)$ is dense in $\prod_{v \in V_{F,\infty}} G(F_v)$. (See theorem 7.7 of [PR].) Suppose further that S is any finite set of places of F and that $T_v \subset G \times \mathrm{Spec}\, F_v$ is a maximal torus for all $v \in S$. Then there is a maximal torus $T \subset G$ such that $T \times \mathrm{Spec}\, F_v$ is $G(F_v)$-conjugate to T_v for all $v \in S$. (See corollary 3 to proposition 7.3 of [PR].)

For any Galois extension E/F (not necessarily finite, but F still a number field) we write $\ker^1(\mathrm{Gal}\,(E/F), G(E))$ for

$$\ker(H^1(\mathrm{Gal}\,(E/F), G(E)) \to \prod_{v \in V_F} H^1(\mathrm{Gal}\,(E_w/F_v), G(E_w))).$$

We will sometimes write $\ker^1(F,G)$ for $\ker^1(\mathrm{Gal}(\overline{F}/F), G(\overline{F}))$. If G is reductive then $\ker^1(\mathrm{Gal}(E/F), G(E))$ is finite. It vanishes if G is semi-simple and either adjoint or simply connected. (See for instance theorems 6.6, 6.19 and 6.22 of [PR].)

Lemma 2.1. *Suppose that $G/\mathbb{Q}$ is a reductive group and $E/\mathbb{Q}$ is a finite totally imaginary Galois extension such that each connected component of $G^{\mathrm{ad}}(\mathbb{R})$ contains a point of $G^{\mathrm{ad}}(\mathbb{Q})_E$. Then*

$$\begin{array}{c}\ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}(E/\mathbb{Q}), G(\mathbb{A}_E^\infty)) \oplus H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))) \\ \downarrow \\ \ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), G(E)) \to H^1(\mathrm{Gal}(E/\mathbb{Q}), G(\mathbb{A}_E)))\end{array}$$

is surjective.

Proof: By the Hasse principle for adjoint semi-simple groups we see that any element of $\ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), G(E)) \to H^1(\mathrm{Gal}(E/\mathbb{Q}), G(\mathbb{A}_E)))$ can be lifted to an element of $\boldsymbol{\zeta} \in \ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}(E/\mathbb{Q}), G(\mathbb{A}_E)))$. Then we have

$$\begin{array}{rcl} G^{\mathrm{ad}}(\mathbb{R})/G(\mathbb{R}) & = & \pi_0(G^{\mathrm{ad}}(\mathbb{R}))/\pi_0(G(\mathbb{R})) \\ & = & \ker(H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C})) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))). \end{array}$$

(The first equality follows from the open mapping theorem.) Choose $\gamma \in G^{\mathrm{ad}}(\mathbb{Q})_E$ lying in a connected component of $G^{\mathrm{ad}}(\mathbb{R})$ which maps to the image of the restriction of $\boldsymbol{\zeta}$. Let $o(\gamma) \in \ker(H^1(\mathrm{Gal}(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}(E/\mathbb{Q}), G(E))$ denote the obstruction to lifting γ to $G(\mathbb{Q})$. (If $\widetilde{\gamma}$ denote a lift of γ to $G(E)$ and let $o(\gamma)$ is represented by the cocycle $\sigma \mapsto \widetilde{\gamma}^\sigma \widetilde{\gamma}^{-1}$.) Then $\boldsymbol{\zeta} o(\gamma)^{-1}$ has the same image as $\boldsymbol{\zeta}$ in $H^1(\mathrm{Gal}(E/\mathbb{Q}), G(E))$ but maps to 0 in $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))$. The lemma follows.□

2.3. **Kottwitz's extensions.** We refer the reader to [ST] for the properties of Kottwitz cohomology which we will require. We present here only a brief summary.

If $E/\mathbb{Q}$ is a finite Galois extension of number fields we will write $T_{2,E}$ (resp. $T_{3,E}$) for the protorus over $\mathbb{Q}$ with cocharacter group $\mathbb{Z}[V_E]$ (resp. $\mathbb{Z}[V_E]_0$) with its natural action of $\mathrm{Gal}(E/\mathbb{Q})$. Thus there is a natural short exact sequence

$$(0) \longrightarrow \mathbb{G}_m \longrightarrow T_{2,E} \longrightarrow T_{3,E} \longrightarrow (0).$$

We will denote by $\pi_w : T_{2,E} \to \mathbb{G}_m$ the character corresponding to $w \in V_E$. We have

$$\prod_{w \in V_E} \pi_w : T_{2,E}(\mathbb{A}_E) \xrightarrow{\sim} \prod_{w \in V_E} \mathbb{A}_E^\times,$$

but with Galois action given by

$$\sigma((x_w)_w) = (\sigma x_{\sigma^{-1}w})_w.$$

If $D \supset E$ are finite Galois extensions of $\mathbb{Q}$ the map

$$\begin{array}{ccc} \mathbb{Z}[V_D] & \longrightarrow & \mathbb{Z}[V_E] \\ \sum_u m_u u & \longmapsto & \sum_u m_u u|_E \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,E} & \longrightarrow & T_{3,E} & \longrightarrow & (0) \\ & & || & & \iota^0_{D/E} \downarrow & & \downarrow \iota^0_{D/E} & & \\ (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,D} & \longrightarrow & T_{3,D} & \longrightarrow & (0); \end{array}$$

and the map

$$\begin{array}{ccc} \mathbb{Z}[V_E] & \longrightarrow & \mathbb{Z}[V_D] \\ \sum_w m_w w & \longmapsto & \sum_u [D_u : E_{u|_E}] m_{u|_E} u \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,D} & \longrightarrow & T_{3,D} & \longrightarrow & (0) \\ & & [D:E] \downarrow & & \eta^0_{D/E} \downarrow & & \downarrow \eta^0_{D/E} & & \\ (0) & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{2,E} & \longrightarrow & T_{3,E} & \longrightarrow & (0), \end{array}$$

and

$$\eta^0_{D/E} \circ \iota^0_{D/E} = [D : E].$$

We set

$$\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 = \prod_{w \in V_E} E_w^\times \subset T_{2,E}(\mathbb{A}_E)$$

(with $E_w^\times$ thought of inside the w-copy of $\mathbb{A}_E^\times$) and

$$\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 = \{(x_w) \in T_{2,E}(\mathbb{A}_E) :\ x_w \bmod E^\times \text{ is independent of } w\} \subset T_{2,E}(\mathbb{A}_E).$$

These are preserved by $\mathrm{Gal}\,(E/\mathbb{Q})$.

In [ST] we defined abelian groups $\mathcal{Z}(E/\mathbb{Q}) \supset \mathcal{B}(E/\mathbb{Q})$ with compatible actions of $T_{2,E}(\mathbb{A}_E)$ such that $T_{2,E}(\mathbb{A}_E)$ acts transitively on the quotient

$$\mathcal{H}(E/\mathbb{Q}) = \mathcal{Z}(E/\mathbb{Q})/\mathcal{B}(E/\mathbb{Q}).$$

The stabilizer in $T_{2,E}(\mathbb{A}_E)$ of any element of $\mathcal{H}(E/\mathbb{Q})$ is $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 T_{2,E}(\mathbb{A})$. To an element $\mathfrak{a} \in \mathcal{H}(E/\mathbb{Q})$ we associate (uniquely up to unique isomorphism):

(1) Extensions

$$(0) \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0)$$

and

$$(0) \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 \longrightarrow \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0).$$

(2) Writing $\mathcal{E}_2(E/\mathbb{Q})_{\mathfrak{a}}$ for the pushout of $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}$ along $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 \hookrightarrow T_{2,E}(\mathbb{A}_E)$, a canonical map of extensions

$$\mathrm{loc}_{\mathfrak{a}} : \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}} \longrightarrow \mathcal{E}_2(E/\mathbb{Q})_{\mathfrak{a}}.$$

(3) An extension

$$(0) \longrightarrow T_{3,E}(E) \longrightarrow \mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0)$$

defined as the pushout of $\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}}$ along $\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 \to T_{3,E}(E)$.

(4) An extension

$$(0) \longrightarrow \mathbb{A}_E^\times/E^\times \longrightarrow W_{E/\mathbb{Q},\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0)$$

defined as the pushout of $\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}}$ along $\mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 \to \mathbb{A}_E^\times/E^\times$. The extension $W_{E/\mathbb{Q},\mathfrak{a}}$ is isomorphic to the Weil group $W_{E^{\mathrm{ab}}/\mathbb{Q}}$. This isomorphism is *not* canonical: it is only defined up to composition with conjugation by an element of $\mathbb{A}_E^\times/E^\times$. (The global Weil group

$$W_{E^{\mathrm{ab}}/\mathbb{Q}} = W_{\overline{\mathbb{Q}}/\mathbb{Q}}/\overline{[W_{\overline{\mathbb{Q}}/E}, W_{\overline{\mathbb{Q}}/E}]},$$

is defined up to an isomorphism that is only unique up to composition with conjugation by an element of $\overline{E^\times(E_\infty^\times)^0}/E^\times$.)

(5) If $S \subset V_{\mathbb{Q}}$, an extension

$$(0) \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S^0 \longrightarrow \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0)$$

defined as the pushout of $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})$ along the projection

$$\pi_S : \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 = \prod_{w \in V_E} E_w^\times \twoheadrightarrow \prod_{w \in V_{E,S}} E_w^\times = \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S^0.$$

We will also write $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}^{V_{\mathbb{Q}}-S} = \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}$.

(6) If $w|v$ are places of E and $\mathbb{Q}$, an extension

$$(0) \longrightarrow E_w^\times \longrightarrow W_{E_w/\mathbb{Q}_v,\mathfrak{a}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q})_w \longrightarrow (0)$$

defined as the pushout of $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}|_{\mathrm{Gal}\,(E/\mathbb{Q})_w}$ along $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 \to E_w^\times$. There is an isomorphism of extensions

$$W_{E_w/\mathbb{Q}_v,\mathfrak{a}} \cong W_{(E\mathbb{Q}_v)^{\mathrm{ab}}/\mathbb{Q}_v},$$

where $W_{(E\mathbb{Q}_v)^{\mathrm{ab}}/\mathbb{Q}_v}$ denotes the local Weil group. This isomorphism is *not* canonical, but only defined up to composition with conjugation by an element of $E_w^\times$. (In this case $W_{(E\mathbb{Q}_v)^{\mathrm{ab}}/\mathbb{Q}_v}$ is defined up to unique isomorphism.)

(7) For $w|v$ are places of E and $\mathbb{Q}$, a map of extensions

$$\iota_w^{\mathfrak{a}} : W_{E_w/\mathbb{Q}_v,\mathfrak{a}} \hookrightarrow W_{E/\mathbb{Q},\mathfrak{a}}$$

compatible with $E_w^\times \hookrightarrow \mathbb{A}_E^\times/E^\times$.

If $\rho : E^{\mathrm{ab}} \hookrightarrow \overline{\mathbb{Q}_v}$, then there is a map of extensions (the 'decomposition group')

$$\theta_\rho : W_{(\rho(E)\mathbb{Q}_v)^{\mathrm{ab}}/\mathbb{Q}_v} \hookrightarrow W_{E^{\mathrm{ab}}/\mathbb{Q}},$$

which is well defined up to composition with conjugation by an element of $\overline{E^\times(E_\infty^\times)^0}/E^\times$. Then θ_ρ and $\iota_w^{\mathfrak{a}}$ will differ (after making the above identifications) by composition with conjugation by an element of $\mathbb{A}_E^\times/E^\times$ (of course depending on ρ).

(Thus the choice of $\mathfrak{a} \in \mathcal{H}(E/\mathbb{Q})$ inter alia gives rise to a preferred decomposition group in $\mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ above each place w of E. We think of the choice of $\mathfrak{a}$ as being analogous to the choice of such decomposition groups.)

Diagramatically we have:

$$\begin{array}{ccccc}
\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}} & \twoheadleftarrow & \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}} & \twoheadrightarrow & W_{E/\mathbb{Q},\mathfrak{a}} \\
 & & \downarrow{\scriptstyle \mathrm{loc}_{\mathfrak{a}}} & & \uparrow \\
\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}} & \hookrightarrow & \mathcal{E}_2(E/\mathbb{Q})_{\mathfrak{a}} & & {\scriptstyle \iota_w^{\mathfrak{a}}} \\
\uparrow & & & & \\
\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}|_{\mathrm{Gal}\,(E/\mathbb{Q})_w} & \twoheadrightarrow & W_{E_w/\mathbb{Q}_v,\mathfrak{a}} & \nearrow &
\end{array}$$

The choice of a 'cocycle' $\boldsymbol{\alpha} \in \mathfrak{a}$ gives rise to distinguished lifts $e^{\mathrm{glob}}_{\boldsymbol{\alpha}}(\sigma) \in \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})_{\mathfrak{a}}$ and $e^{\mathrm{loc}}_{\boldsymbol{\alpha}}(\sigma) \in \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}$ of $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$.

If $t \in T_{2,E}(\mathbb{A}_E)$ there are canonical isomorphisms

$$\mathfrak{z}_t : \mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}} \xrightarrow{\sim} \mathcal{E}^?(E/\mathbb{Q})_{{}^t\mathfrak{a}}$$

for each of the extensions considered above. They commute with all the arrows in the above diagram, except for the arrows that go between the first row and one of the other rows. For these we have

$$\mathfrak{z}_t \circ \mathrm{loc}_{\mathfrak{a}} = \mathrm{conj}_t \circ \mathrm{loc}_{{}^t\mathfrak{a}} \circ \mathfrak{z}_t$$

and

$$\mathrm{conj}_{t_w} \circ \mathfrak{z}_t \circ \iota_w^{\mathfrak{a}} = \iota_w^{{}^t\mathfrak{a}} \circ \mathfrak{z}_t.$$

More generally, if $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, if $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$ and $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ we can find $t \in T_{2,E}(\mathbb{A}_D)$ with, in the notation of [ST],

$${}^t \inf_{D/E} \mathfrak{a}_E^+ = \eta_{D/E,*}\mathfrak{a}_D^+.$$

The element t is unique up to

$$\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0_D \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0_D T_{2,\mathbb{Q}}(\mathbb{A}) \prod_{w \in V_E} \overline{E^\times (E_\infty^\times)^0}.$$

(Again using the notation of [ST].) If $C \supset D$ is another finite Galois extension of $\mathbb{Q}$ and if $\mathfrak{a}_C^+ \in \mathcal{H}(C/F)^+$ satisfies $\eta_{C/D,*}\mathfrak{a}_C^+ = {}^{t'} \inf_{C/D} \mathfrak{a}_D^+$ with $t' \in T_{2,D}(\mathbb{A}_C)$, then

$$\eta_{C/E,*}\mathfrak{a}_C^+ = {}^{t\eta_{D/E}(t')} \inf_{C/E} \mathfrak{a}_E^+.$$

We define the following pointed sets of algebraic cocycles:

(1) If G/F is a linear algebraic group, we define $Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ to be the set of 1-cocycles $\phi : \mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}} \to G(E)$ such that there is a, necessarily unique, algebraic homomorphism $\boldsymbol{\nu}_\phi : T_{3,E} \to Z(G)$ over $\mathbb{Q}$ with $\phi|_{T_{3,E}(E)} =$

$\boldsymbol{\nu}_\phi$. This pointed set of cocycles has a natural action of $G(E)$ via the usual coboundary map and we will denote the quotient $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$.

(2) If $G/\mathbb{A}_S$ is a linear algebraic group, we define $Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}}$ to be the set of 1-cocycles $\phi : \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}} \to G(\mathbb{A}_{E,S})$ such that there are for each $w \in V_{E,S}$, necessarily unique, algebraic homomorphisms $\boldsymbol{\nu}_{\phi,w} : \mathbb{G}_m \to Z(G)$ over E_w, almost all of which are trivial, with $\phi|_{\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0_S} = \prod_{w \in V^S_E} \boldsymbol{\nu}_{\phi,w}$. This pointed set of cocyles has a natural actions of $G(\mathbb{A}_{E,S})$ via the usual coboundary map and we will denote the quotient $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}}$.

(3) If $G/\mathbb{Q}_v$ is a linear algebraic group, we define $Z^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}$ to be the set of 1-cocycles $\phi : W_{E_w/\mathbb{Q}_v,\mathfrak{a}} \to G(E_w)$ for which there is a necessarily unique, algebraic homomorphisms $\boldsymbol{\nu}_\phi : \mathbb{G}_m \to Z(G)$ over $\mathbb{Q}_v$ with $\phi|_{E_w^\times} = \boldsymbol{\nu}_\phi$. This pointed set of cocyles has a natural actions of $G(E_w)$ via the usual coboundary map, and we will denote the quotient $H^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}$.

There are natural equivariant maps

$$\mathrm{loc}_{\mathfrak{a}} : Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}}$$

and, for $S \subset S'$,

$$\mathrm{res}_S : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S',\mathfrak{a}}, G(\mathbb{A}_{E,S'}))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}}$$

and for $w \in V_{E,S}$,

$$\mathrm{res}_w : Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}.$$

These maps are functorial in G and the maps res_S and res_w are compatible in triples whenever this makes sense. They give an isomorphism

$$Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}, G(\mathbb{A}_{E,S}))_{\mathrm{basic}} \xrightarrow{\sim} {\prod_{v\in S}}' Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\{v\},\mathfrak{a}}, G(E_v))_{\mathrm{basic}}$$

where the products are restricted with respect to the subsets $Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(\mathcal{O}_{E,v}))$.

To be more concise we will sometimes write $\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}$ either $\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}$ or $\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{S,\mathfrak{a}}$ or $W_{E_w/\mathbb{Q}_v,\mathfrak{a}}$ and $A_{\mathbb{Q}}$ for $\mathbb{Q}$ (resp. $\mathbb{A}_S$, resp. $\mathbb{Q}_v$) and A_E for E (resp. $\mathbb{A}_{E,S}$, resp. E_w) and $\mathrm{Gal}^?(E/\mathbb{Q})$ for $\mathrm{Gal}\,(E/\mathbb{Q})$ (resp. $\mathrm{Gal}\,(E/\mathbb{Q})$, resp. $\mathrm{Gal}\,(E_w/\mathbb{Q}_v)$).

If $t \in T_{2,E}(\mathbb{A}_E)$ then $\mathfrak{z}_t^{-1}$ induces maps

$$z_t : Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}} \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{{}^t\mathfrak{a}}, G(A_E))_{\mathrm{basic}}$$

which are functorial in G and are equivariant for the $G(A_E)$-action, so that they pass to cohomology. We have $\boldsymbol{\nu} \circ z_t = \boldsymbol{\nu}$ and $z_{t_1 t_2} = z_{t_1} \circ z_{t_2}$. The maps z_t commute with the maps res^S and res_w. We have

$$(\mathrm{loc}_{{}^t\mathfrak{a}} \circ z_t)(\phi) = {}^{\boldsymbol{\nu}_\phi(t)}(z_t \circ \mathrm{loc}_{\mathfrak{a}})(\phi).$$

The maps the z_t induce in cohomology are independent of t, and so $H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}}$ together with the maps $\mathrm{loc}_{\mathfrak{a}}$, res_S and res_w on cohomology are canonically independent of $\mathfrak{a}$. Thus we will denote it simply $H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), G(A_E))_{\mathrm{basic}}$, loc, res_S and

res_w. We get isomorphisms

$$\begin{array}{rcl} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S, G(\mathbb{A}_{E,S}))_{\mathrm{basic}} & \stackrel{\sim}{\longrightarrow} & \prod'_{v\in S} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\{v\}}, G(E_v))_{\mathrm{basic}} \\ & \stackrel{\sim}{\longrightarrow} & \prod'_{v\in S} H^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v}, G(E_w))_{\mathrm{basic}} \end{array}$$

where for each $v \in S$ we choose a place w of E above it, and where the products are restricted with respect to the subsets $H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(\mathcal{O}_{E,v}))$ (resp. $H^1(\mathrm{Gal}\,(E_w/\mathbb{Q}_v), G(\mathcal{O}_{E,w}))$). The kernel of the map

$$\mathrm{loc} : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$$

equals $\ker^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$.

If $E^0_v/\mathbb{Q}_v$ is a finite Galois extension and $G/\mathbb{Q}_v$ is an algebraic group, we define $Z^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}}$ to be the subset of $Z^1(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))$ consisting of cocycles ϕ whose restriction to $W_{(E^0_v)^{\mathrm{ab}}/E^0_v}$ are of the form $\boldsymbol{\nu}_\phi \circ \mathrm{Art}_{E^0_v}^{-1}$ for some $\boldsymbol{\nu}_\phi \in X_*(Z(G))(\mathbb{Q}_v)$. The pointed set $Z^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}}$ is preserved by the coboundary action of $G(E^0_v)$ and we denote the quotient $H^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}}$. If $E/\mathbb{Q}$ is a finite Galois extension and $w|v$ is a place of E such that $E_w \cong E^0_v$ over $\mathbb{Q}_v$, then the choice of an isomorphism of extensions $W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v} \cong W_{E_w/\mathbb{Q}_v,\mathfrak{a}}$ gives rise to bijections

$$Z^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}} \cong Z^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}$$

and

$$H^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}} \cong H^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v}, G(E_w))_{\mathrm{basic}},$$

the latter being independent of the choices of isomorphisms $E^0_v \cong E_w$ and $W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v} \cong W_{E_w/\mathbb{Q}_v,\mathfrak{a}}$. The composite of this map with res_w gives a map

$$\mathrm{res}_{E^0_v/\mathbb{Q}_v} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_{E,S}))_{\mathrm{basic}} \longrightarrow H^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}}$$

which is independent of all choices, including the choice of w.

If $E/\mathbb{Q}$ is a finite Galois extension and if for each place v of $\mathbb{Q}$ we fix a finite Galois extension $E^0_v/\mathbb{Q}_v$ isomorphic to $E_w/\mathbb{Q}_v$ for any (and hence every) place w of E above v, then we obtain an identification

$$\prod_{v\in S} \mathrm{res}_{E^0_v/\mathbb{Q}_v} : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S, G(\mathbb{A}_{E,S}))_{\mathrm{basic}} \stackrel{\sim}{\longrightarrow} {\prod_{v\in S}}' H^1_{\mathrm{alg}}(W_{(E^0_v)^{\mathrm{ab}}/\mathbb{Q}_v}, G(E^0_v))_{\mathrm{basic}},$$

where the product is restricted with respect to the subsets $H^1(\mathrm{Gal}\,(E^0_v/\mathbb{Q}_v), G(\mathcal{O}_{E^0_v}))$.

For $G/A_{\mathbb{Q}}$ an algebraic group there is a natural map

$$Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}} \longrightarrow Z^1(\mathrm{Gal}^?(E/\mathbb{Q}), G^{\mathrm{ad}}(A_E)).$$

Thus if $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}}$ there is a canonically defined inner form ${}^\phi G$ of G over $A_{\mathbb{Q}}$, together with an isomorphism $\iota_\phi : G \times A_E \stackrel{\sim}{\to} {}^\phi G \times A_E$ such that $\sigma\iota_\phi(g) = \iota_\phi((\mathrm{ad}\,\phi(\sigma))(\sigma g))$ for all $\sigma \in \mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}$ and $g \in G(A_E)$. If $h \in G(A_E)$ then there is a unique isomorphism $\iota_h : {}^\phi G \stackrel{\sim}{\to} {}^{h\phi}G$ over $A_{\mathbb{Q}}$ such that $\iota_h \circ \iota_\phi = \iota_{h\phi} \circ \mathrm{conj}_h$.

If $\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, ({}^{\phi}G)(A_E))_{\mathrm{basic}}$, then $\psi\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}}$ and this gives a bijection of sets

$$Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, ({}^{\phi}G)(A_E))_{\mathrm{basic}} \xrightarrow{\sim} Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}}, G(A_E))_{\mathrm{basic}},$$

but this map does not preserve neutral elements. This product is functorial in G and commutes with $\mathrm{loc}_{\mathfrak{a}}$, res^S and res_w. We have $\boldsymbol{\nu}_{\psi\phi} = \boldsymbol{\nu}_{\psi}\boldsymbol{\nu}_{\psi}$ and $z_t(\psi\phi) = z_t(\psi)z_t(\phi)$. The composite

$$\iota_{\psi\phi} \circ \iota_{\phi}^{-1} \circ \iota_{\psi}^{-1} : {}^{\psi}({}^{\phi}G) \xrightarrow{\sim} {}^{\psi\phi}G$$

is defined over $A_{\mathbb{Q}}$. We have $({}^{g}\psi)\phi = {}^{g}(\psi\phi)$ and so we get a bijection

$$H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), ({}^{\phi}G)(A_E))_{\mathrm{basic}} \xrightarrow{\sim} H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), G(A_E))_{\mathrm{basic}}.$$

Moreover $(\iota_g \circ \psi)^{g}\phi = {}^{g}(\psi\phi)$, and if we use ι_g to identify ${}^{\phi}G$ and ${}^{g\phi}G$ then the map induced in cohomology by ϕ only depends on $[\phi] \in H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), G(A_E))_{\mathrm{basic}}$.

If $\boldsymbol{\phi} \in H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), G(A_E))_{\mathrm{basic}}$ we will sometimes write ${}^{\boldsymbol{\phi}}G/A_{\mathbb{Q}}$ for ${}^{\phi}G$ for any $\phi \in \boldsymbol{\phi}$. However we must keep in mind that ${}^{\boldsymbol{\phi}}G$ is unique up to an isomorphism, that is only unique up to composition with conjugation by an element of ${}^{\boldsymbol{\phi}}G(A_{\mathbb{Q}})$.

Now suppose that $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ and $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D^+ = {}^{t}\inf_{D/E}\mathfrak{a}_E^+$. Then there is a map

$$\inf_{D/E,t} : Z^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q})_{\mathfrak{a}_E}, G(A_E))_{\mathrm{basic}} \longrightarrow Z^1_{\mathrm{alg}}(\mathcal{E}^?(D/\mathbb{Q})_{\mathfrak{a}_D}, G(A_D))_{\mathrm{basic}}.$$

This map is functorial in G, commutes with products, and passes to cohomology. These maps are all injective even on the level of cohomology. They commute with the maps res^S and res_w. We have

$$\mathrm{loc}_{\mathfrak{a}_D}(\inf_{D/E,t}(\phi)) = {}^{\boldsymbol{\nu}_{\phi}(t)}\inf_{D/E,t}(\mathrm{loc}_{\mathfrak{a}_E}\phi).$$

Note that $\inf_{E/E,t} = z_t$ and $\boldsymbol{\nu}_{\inf_{D/E,t}\phi} = \boldsymbol{\nu}_{\phi} \circ \eta_{D/E}$. If $C \supset D$ is another finite Galois extension of $\mathbb{Q}$ and if $\mathfrak{a}_C \in \mathcal{H}(C/\mathbb{Q})$ and if $t' \in T_{2,D}(\mathbb{A}_C)$ with $\eta_{C/D,*}\mathfrak{a}_C = {}^{t'}\inf_{C/D}\mathfrak{a}_D$, then

$$\inf_{C/D,t'} \circ \inf_{D/E,t} = \inf_{C/E,t\eta_{D/E}(t')}.$$

Suppose that $a \in T_{2,E}(\mathbb{A})$ and $b \in \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0_D$ and $c \in \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0_D$. If $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ then

$$\inf_{D/E,abct}(\phi) = {}^{\boldsymbol{\nu}_{\phi}(b)}\inf_{D/E,t}(\phi);$$

if $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S_{\mathfrak{a}}, G(\mathbb{A}^S_E))_{\mathrm{basic}}$ then

$$\inf_{D/E,abct}(\phi) = {}^{\boldsymbol{\nu}_{\phi}(\pi^S(c)^{-1})}\inf_{D/E,t}(\phi);$$

and if $\phi \in Z^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v,\mathfrak{a}}, G(E_w))_{\mathrm{basic}}$ then

$$\inf_{D/E,abct}(\phi) = {}^{\boldsymbol{\nu}_\phi(\pi_w(c)^{-1})} \inf_{D/E,t}(\phi).$$

Thus the maps

$$\inf_{D/E,t} : H^1_{\mathrm{alg}}(\mathcal{E}^?(E/\mathbb{Q}), G(A_E))_{\mathrm{basic}} \longrightarrow H^1_{\mathrm{alg}}(\mathcal{E}^?(D/\mathbb{Q}), G(A_D))_{\mathrm{basic}}$$

are independent of t, and so we will denote them simply $\inf_{D/E}$. They commute with loc, res^S and res_w, and we have $\inf_{C/D} \circ \inf_{D/E} = \inf_{C/E}$. Following Kottwitz, we define

$$B(\mathbb{Q}, G)_{\mathrm{basic}} = \lim_{\to, E} H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$$

and

$$B^{\mathrm{loc}}(\mathbb{Q}, G)^S_{\mathrm{basic}} = \lim_{\to, E} H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^S, G(\mathbb{A}_E^S))_{\mathrm{basic}}$$

and

$$B(\mathbb{Q}_v, G)_{\mathrm{basic}} = \lim_{\to, E_w} H^1_{\mathrm{alg}}(W_{E_w/\mathbb{Q}_v}, G(E_w))_{\mathrm{basic}}.$$

If $G/\mathbb{Q}$ is reductive and split by E, Kottwitz defines maps

$$\kappa_S : H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S, G(\mathbb{A}_E))_{\mathrm{basic}} \longrightarrow (\Lambda_G \otimes \mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})}$$

(which we will denote simply κ if $S = V_{\mathbb{Q}}$) and

$$\kappa : H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}} \longrightarrow (\Lambda_G \otimes \mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})}$$

with the following properties:

(1) If $\boldsymbol{\phi} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$, then $(\kappa \circ \mathrm{loc})(\boldsymbol{\phi})$ equals the image of $\kappa(\boldsymbol{\phi})$ under $(\Lambda_G \otimes \mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})} \to (\Lambda_G \otimes \mathbb{Z}[V_E])_{\mathrm{Gal}\,(E/\mathbb{Q})}$.

(2) If $f : G \to G'$, then $\kappa \circ f_* = (f \otimes 1) \circ \kappa$, and similarly for κ_S.

(3) $\kappa \circ \mathrm{res}_S$ equals κ composed with $(\Lambda_G \otimes \mathbb{Z}[V_{E,S'}])_{\mathrm{Gal}\,(E/\mathbb{Q})} \twoheadrightarrow (\Lambda_G \otimes \mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})}$.

(4) $\kappa(\boldsymbol{\psi}\boldsymbol{\phi}) = \kappa(\boldsymbol{\psi})\kappa(\boldsymbol{\phi})$.

(5) $\kappa_S \circ \inf_{D/E}$ equals the composition of κ and the inverse of the natural isomorphism $(\Lambda_G \otimes \mathbb{Z}[V_{D,S}])_{\mathrm{Gal}\,(D/\mathbb{Q})} \xrightarrow{\sim} (\Lambda_G \otimes \mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})}$. Thus we obtain maps

$$\kappa_S : B^{\mathrm{loc}}(\mathbb{Q}, G)_{S,\mathrm{basic}} \longrightarrow (\Lambda_G \otimes \mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})}.$$

(6) $\kappa \circ \inf_{D/E}$ equals the composition of κ and the inverse of the natural isomorphism $(\Lambda_G \otimes \mathbb{Z}[V_D]_0)_{\mathrm{Gal}\,(D/\mathbb{Q})} \xrightarrow{\sim} (\Lambda_G \otimes \mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})}$. Thus we obtain a map

$$\kappa : B(\mathbb{Q}, G)_{\mathrm{basic}} \longrightarrow (\Lambda_G \otimes \mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})}.$$

This map has finite fibres.

(7) If S consists of finite places then

$$\kappa_S : B^{\mathrm{loc}}(\mathbb{Q}, G)_{S,\mathrm{basic}} \xrightarrow{\sim} (\Lambda_G \otimes \mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})}$$

is an isomorphism.

(8) If S is a set of places of $\mathbb{Q}$ we will write $B(\mathbb{Q},G)_{S,\mathrm{basic}}$ for the preimage in $B(\mathbb{Q},G)_{\mathrm{basic}}$ of the image of $\Lambda_G\otimes\mathbb{Z}[V_{E,S}]_0$ in $(\Lambda_G\otimes\mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})}$. If S is finite then there exists a finite extension D/E Galois over $\mathbb{Q}$ so that $B(\mathbb{Q},G)_{S,\mathrm{basic}}$ is contained in the image of

$$H^1_{\mathrm{alg}}(\mathcal{E}_3(D/\mathbb{Q}),G(D))_{\mathrm{basic}}\longrightarrow B(\mathbb{Q},G)_{\mathrm{basic}}.$$

(9) There is a cartesian square

$$\begin{array}{ccc} B(\mathbb{Q},G)_{\mathrm{basic}} & \stackrel{\mathrm{res}_w\circ\mathrm{loc}}{\longrightarrow} & B(\mathbb{R},G)_{\mathrm{basic}} \\ \kappa\downarrow & & \kappa\downarrow \\ (\Lambda_G\otimes\mathbb{Z}[V_E]_0)_{\mathrm{Gal}\,(E/\mathbb{Q})} & \longrightarrow & \Lambda_{G,\mathrm{Gal}\,(E_w/\mathbb{R})} \\ & & \\ \sum_w\lambda_w\otimes w & \longmapsto & \sum_{\sigma\in\mathrm{Gal}\,(E_w/\mathbb{R})\backslash\mathrm{Gal}\,(E/\mathbb{Q})}\sigma\lambda_{\sigma^{-1}w}, \end{array}$$

where $w|\infty$ is any infinite place of E.

We will write

$$\overline{\kappa}_S:H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_S,G(\mathbb{A}_E))_{\mathrm{basic}}\longrightarrow\Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})}$$

for the composition of κ_S with the map

$$\begin{array}{ccc} (\Lambda_G\otimes\mathbb{Z}[V_{E,S}])_{\mathrm{Gal}\,(E/\mathbb{Q})} & \longrightarrow & \Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})} \\ \sum\lambda_w\otimes w & \longmapsto & \sum_w\lambda_w. \end{array}$$

As $\overline{\kappa}_S\circ\inf_{D/E}=\overline{\kappa}_S$ we see that we get a map $\overline{\kappa}_S:B^{\mathrm{loc}}(\mathbb{Q},G)_{S,\mathrm{basic}}\to\Lambda_{G,\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})}$. In the case $S=V_{\mathbb{Q}}$ we will write simply $\overline{\kappa}$, and we have $\overline{\kappa}\circ\mathrm{loc}$ is trivial on $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}),G(E))_{\mathrm{basic}}$. The map

$$\mathrm{loc}:B(\mathbb{Q},G)_{\mathrm{basic}}\twoheadrightarrow\ker\overline{\kappa}\subset B^{\mathrm{loc}}(\mathbb{Q},G)_{\mathrm{basic}}$$

is surjective with finite fibres.

2.4. **Real groups.** *In this section we suppose that $G/\mathbb{R}$ is a reductive group.*

The group of real points $G(\mathbb{R})$ has finitely many connected components. If either G is simply connected semi-simple or $G(\mathbb{R})$ is compact, then $G(\mathbb{R})$ is connected. (See theorem 3.6 and its first corollary and proposition 7.6 of [PR].) If $G/\mathbb{R}$ is a reductive group and H is a normal subgroup defined over $\mathbb{R}$ then the image of $G(\mathbb{R})\to(G/H)(\mathbb{R})$ is a union of connected components. (The image is open by the open mapping theorem.) If $(G/H)(\mathbb{R})$ is connected, for example if it is compact, then $G(\mathbb{R})\twoheadrightarrow(G/H)(\mathbb{R})$. We will write $G(\mathbb{R})^+$ for the connected component of the identity in $G(\mathbb{R})$ in the archimedean topology. Because $G(\mathbb{R})$ is Zariski dense in G, we see that $Z_G(\mathbb{R})=Z_G(G(\mathbb{R}))$ and that $G(\mathbb{R})^{\mathrm{ad}}$ naturally embeds in $G^{\mathrm{ad}}(\mathbb{R})$ (and $G(\mathbb{R})^{\mathrm{ad}}\supset G^{\mathrm{ad}}(\mathbb{R})^+$.)

If $G/\mathbb{R}$ is reductive then a maximal torus $T\subset G_{/\mathbb{R}}$ is called *fundamental* if its split rank is minimal among the split ranks of all maximal tori. All fundamental maximal tori are conjugate by $G(\mathbb{R})$. (See [BW] section I.7.1.) If G' is an inner form of G, then fundamental maximal tori in G and G' are isomorphic. (See lemma 2.8 of [Sh].)

If T is a fundamental torus and if $T^{\mathrm{ad}}(\mathbb{R})$ is compact (or equivalently if c acts by -1 on $X_*(T^{\mathrm{ad}})$) then $W_T(\mathbb{R}) = W_T(\mathbb{C})$. Moreover if T is a fundamental torus, then $T^{\mathrm{ad}}(\mathbb{R})$ is compact if and only if G has an inner form G' with $G'^{,\mathrm{ad}}(\mathbb{R})$ compact. (See proposition 3 of [LS].)

If $G^{\mathrm{ad}}(\mathbb{R})$ is compact then all maximal tori T are fundamental, and hence conjugate. Moreover, in this case, $W_{T,\mathbb{R}} = W_T(\mathbb{R}) = W_T(\mathbb{C})$, so that any two embeddings $i, i' : T \hookrightarrow G$ are conjugate under $G(\mathbb{R})$. (In the case that $G(\mathbb{R})$ is compact the equality $W_{T,\mathbb{R}} = W_T(\mathbb{R})$ is well known, see for instance theorem 11.36 of [H]. The more general case $G^{\mathrm{ad}}(\mathbb{R})$ compact reduces to this because $G(\mathbb{R}) \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})$.)

If $\mu \in X_*(G)(\mathbb{C})$ then we will call μ *basic* if $\mu^c\mu$ factors through $Z(G)$. In this case ${}^c\mu\mu = \mu^c\mu$. We will write $X_*(G)(\mathbb{C})_{\mathrm{basic}}$ for the set of basic cocharacters. Being basic is preserved under $G(\mathbb{R})$-conjugacy. If μ is basic, then μ factors through a fundamental maximal torus. (To see this work in G^{ad}. Then $(\mathrm{Im}\,\mu)(\mathbb{R})$ is compact and so contained in some maximal compact subgroup of $G^{\mathrm{ad}}(\mathbb{R})$. Hence it is contained in a maximal compact torus, and so in a fundamental torus. See section I.7.1 of [BW].) If $G^{\mathrm{ad}}(\mathbb{R})$ is compact and $\mu \in X_*(G)$ factors through a torus defined over $\mathbb{R}$, then it factors through a maximal torus T defined over $\mathbb{R}$ and is basic (because c acts on $X_*(T^{\mathrm{ad}})$ by -1).

If $\mu \in X_*(G)(\mathbb{C})$ then we will call μ *compactifying* if μ is basic and $\mathrm{ad}\,\mu(-1) \in G^{\mathrm{ad}}(\mathbb{R})$ is a Cartan involution (i.e. $G^{\mathrm{ad}}(\mathbb{C})^{\mathrm{conj}_{\mu(-1)} \circ c = 1}$ is compact). (See for instance section 2 of [BC] for basic facts about Cartan involutions.) Being compactifying is preserved under $G(\mathbb{R})$-conjugacy. If G admits a compactifying cocharacter, then G^{ad} has a compact inner form.

If Y is a $G(\mathbb{R})$ conjugacy class of elements of $X_*(G)$ we will call it basic (resp. compactifying) if it contains an element which is basic (resp. compactifying), in which case all its elements are basic (resp. compactifying). If Y is basic (resp. compactifying) so is $Y^{-1} = \{\mu^{-1} : \mu \in Y\}$. If Y is basic we will write $\boldsymbol{\nu}_Y = \mu^c\mu \in X_*(Z(G))$ for any $\mu \in Y$. (This is of course independent of the choice of $\mu \in Y$.)

Lemma 2.2. *Suppose that $G/\mathbb{R}$ is a reductive group and that $G^{\mathrm{ad}}(\mathbb{R})$ is compact. Any G-conjugacy class $C \subset X_*(G)$ contains a unique $G(\mathbb{R})$-conjugacy class $Y(C)_G$ consisting of those cocharacters in $C(\mathbb{C})$ which factor through a maximal torus defined over $\mathbb{R}$. The elements of $Y(C)_G$ are in fact basic.*

Proof: Any $\mu \in C(\mathbb{C})$ factors through some maximal torus and hence is conjugate to a cocharacter factoring through any other maximal torus, for instance one defined over $\mathbb{R}$. If $\mu, \mu' \in C(\mathbb{C})$ factor through maximal tori defined over $\mathbb{R}$, then replacing μ' by a $G(\mathbb{R})$-conjugate, we may assume it factors through the same maximal torus T (defined over $\mathbb{R}$) as μ. Then μ and μ' are conjugate by an element of $W_T(\mathbb{C}) = W_{T,\mathbb{R}}$, i.e. μ and μ' are $N_G(T)(\mathbb{R})$-conjugate. $\square$

Choose a representative $\alpha^0_{\mathbb{C}/\mathbb{R}}$ for the canonical class $[\alpha_{\mathbb{C}/\mathbb{R}}] \in H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$ defined by

$$\alpha^0_{\mathbb{C}/\mathbb{R}}(\sigma_1, \sigma_2) = \begin{cases} -1 & \text{if } \sigma_1 = \sigma_2 = c \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$W_{\mathbb{C}/\mathbb{R}, \alpha^0_{\mathbb{C}/\mathbb{R}}} \cong \langle \mathbb{C}^\times, j : \ j^2 = -1 \text{ and } jzj^{-1} = {}^c z \rangle,$$

with section $e_{\alpha^0_{\mathbb{C}/\mathbb{R}}}(1) = 1$ and $e_{\alpha^0_{\mathbb{C}/\mathbb{R}}}(c) = j$. Thus an element of $Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}, \alpha^0_{\mathbb{C}/\mathbb{R}}}, G(\mathbb{C}))_{\mathrm{basic}}$ is a pair (ν, J) where $\nu \in X_*(Z(G))(\mathbb{R})$ and $J \in G(\mathbb{C})$ satisfy

$$J^c J = \nu(-1).$$

Moreover $[(\nu, J)] = [(\nu', J')] \in H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))_{\mathrm{basic}}$ if and only if $\nu = \nu'$ and there exists $g \in G(\mathbb{C})$ such that

$$J' = g J^c g^{-1}.$$

If $\mu \in X_*(G)$ is basic, then we obtain an element $\widehat{\lambda}_G(\mu) \in Z^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}, \alpha^0_{\mathbb{C}/\mathbb{R}}}, G(\mathbb{C}))$ defined by

$$\widehat{\lambda}_G(\mu) = (\mu^c \mu, \mu(-1)).$$

We have that $\kappa(\widehat{\lambda}_G(\mu)) = \lambda_G(\mu)$ and $\boldsymbol{\nu}_{\widehat{\lambda}_G(\mu)} = \mu^c \mu$. This induces a surjective map

$$\widehat{\boldsymbol{\lambda}}_G : G(\mathbb{R}) \backslash X_*(G)(\mathbb{C})_{\mathrm{basic}} \twoheadrightarrow H^1_{\mathrm{alg}}(W_{\mathbb{C}/\mathbb{R}}, G(\mathbb{C}))_{\mathrm{basic}}.$$

The image $\widehat{\boldsymbol{\lambda}}_G(\mu)$ depends only on the $G(\mathbb{R})$-conjugacy class of μ, so we will sometimes write $\widehat{\boldsymbol{\lambda}}_G([\mu]_{G(\mathbb{R})})$. For this see section 5.3 of [ST].

If $\mu \in X_*(G)(\mathbb{C})$ is basic, then $\mu \in X_*({}^{\widehat{\lambda}_G(\mu)}G)(\mathbb{C})$ is also basic. If $g \in G(\mathbb{R})$ then $\widehat{\lambda}_G(\mathrm{conj}_g \circ \mu) = {}^g \widehat{\lambda}_G(\mu)$ and

$$X_*(\mathrm{conj}_g) : X_*({}^{\widehat{\lambda}_G(\mu)}G) \xrightarrow{\sim} X_*({}^{\widehat{\lambda}_G(\mathrm{conj}_g \circ \mu)}G)$$

takes μ to $\mathrm{conj}_g \circ \mu$. If Y is a basic $G(\mathbb{R})$-conjugacy class of cocharacters, then the inner form ${}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G$ comes with a canonical cocharacter $\mu_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}$ (equal to the cocharacter μ of ${}^{\widehat{\lambda}_G(\mu)}G$) and the pair $({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G, \mu_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G})$ is unique up to an isomorphism, unique up to composition with conjugation by an element of $Z_{({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G)(\mathbb{R})}(\mu_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G})$. Note that

$$({}^{\widehat{\boldsymbol{\lambda}}_G(Y^{-1})}G, \mu_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y^{-1})}G}) = ({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G, \mu^{-1}_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G})$$

(as canonically as the two sides are defined).

This implies that the group ${}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G$ (which is defined up to an isomorphism unique up to composition with conjugation by an element of $({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G)(\mathbb{R})$) has a canonical basic $({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G)(\mathbb{R})$-conjugacy class of cocharacters $Y_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}$. More precisely if $\phi \in \widehat{\boldsymbol{\lambda}}_G(Y)$ then ${}^\phi G$ has a well defined basic $({}^\phi G)(\mathbb{R})$ conjugacy class $Y_{{}^\phi G}$; and $Y_{{}^{g\phi}G} = \mathrm{conj}_g Y_{{}^\phi G}$. Note that $\widehat{\boldsymbol{\lambda}}_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}(Y^{-1}_{{}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G}) = \widehat{\boldsymbol{\lambda}}_G(Y)^{-1}$.

Suppose that $G^{\mathrm{ad}}(\mathbb{R})$ is compact and that C is a G-conjugacy class of cocharacters of G. If $\phi \in \widehat{\boldsymbol{\lambda}}_G(Y(C)_G^{-1})$ then ${}^\phi G$ comes with a canonical basic $({}^\phi G)(\mathbb{R})$ conjugacy class of cocharacters $Y(C)_{\phi G} = (Y(C)_G^{-1})^{-1}_{\phi G}$; and $Y(C)_{{}^g\phi G} = \mathrm{conj}_g Y(C)_{\phi G}$. We have

$$\widehat{\boldsymbol{\lambda}}_{\phi G}(Y(C)_{\phi G}) = \widehat{\boldsymbol{\lambda}}_G(Y(C)_G^{-1})^{-1}$$

and

$$\kappa(\widehat{\boldsymbol{\lambda}}_G(Y(C)_G^{-1})) = \lambda_G(C)^{-1}$$

and

$$\boldsymbol{\nu}_{\widehat{\boldsymbol{\lambda}}_G(Y(C)_G^{-1})} = \boldsymbol{\nu}^{-1}_{Y(C)_G}.$$

Now suppose that $Y \subset X_*(G)$ is a compactifying $G(\mathbb{R})$-conjugacy class (no longer assuming $G^{\mathrm{ad}}(\mathbb{R})$ is compact) and that C is a G-conjugacy class in $X_*(G)$. Then C is canonically a ${}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G$-conjugacy class in $X_*({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G)$, and so we have

$$\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y)G}(Y(C)^{-1}_{\widehat{\boldsymbol{\lambda}}_G(Y)G}) \in H^1_{\mathrm{alg}}(W_{\mathbb{R}}, {}^{\widehat{\lambda}_G(Y)}G)_{\mathrm{basic}}.$$

We set

$$\widehat{\boldsymbol{\lambda}}_G(Y-C) = \widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y)G}(Y(C)^{-1}_{\widehat{\boldsymbol{\lambda}}_G(Y)G})\widehat{\boldsymbol{\lambda}}_G(Y) \in H^1_{\mathrm{alg}}(W_{\mathbb{R}}, G)_{\mathrm{basic}}.$$

The group ${}^{\widehat{\boldsymbol{\lambda}}_G(Y-C)}G$ comes with a ${}^{\widehat{\boldsymbol{\lambda}}_G(Y-C)}G(\mathbb{R})$-conjugacy class of cocharacters

$$Y(C)_{\widehat{\boldsymbol{\lambda}}_G(Y-C)G} = Y(C)_{\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y)G}(Y(C)^{-1}_{\widehat{\boldsymbol{\lambda}}_G(Y)G})}({}^{\widehat{\boldsymbol{\lambda}}_G(Y)}G).$$

More precisely if $\phi \in \widehat{\boldsymbol{\lambda}}_G(Y-C)$ then ${}^\phi G$ comes with a canonical compactifying $({}^\phi G)(\mathbb{R})$ conjugacy class of cocharacters $Y(C)_{\phi G}$; and $Y(C)_{{}^g\phi G} = \mathrm{conj}_g Y(C)_{\phi G}$. Note that

$$\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y-C)G}(Y(C)_{\widehat{\boldsymbol{\lambda}}_G(Y-C)G}) = \widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y)G}(Y(C)^{-1}_{\widehat{\boldsymbol{\lambda}}_G(Y)G})^{-1}$$

and

$$\kappa_G(\widehat{\boldsymbol{\lambda}}_G(Y-C)) = \lambda_G(Y)/\lambda_G(C)$$

and

$$\boldsymbol{\nu}_{\widehat{\boldsymbol{\lambda}}_G(Y-C)} = \boldsymbol{\nu}_Y/\boldsymbol{\nu}_{Y(C)_{\widehat{\boldsymbol{\lambda}}_G(Y)G}}.$$

If $G^{\mathrm{ad}}(\mathbb{R})$ is compact and C_1, C_2 are G-conjugacy classes of cocharacters, then

$$\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1})G}(C_2 - Y(C_1)_{\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1})G})\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1}) = \widehat{\boldsymbol{\lambda}}_G(Y(C_2)_G^{-1})$$

and

$$Y(C_2)_{\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1})G}(C_2 - Y(C_1)_{\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1})G})}({}^{\widehat{\boldsymbol{\lambda}}_G(Y(C_1)_G^{-1})}G) = Y(C_2)_{\widehat{\boldsymbol{\lambda}}_G(Y(C_2)_G^{-1})G}.$$

2.5. **Some important Kottwitz cohomology classes.** In this section suppose that $G/\mathbb{Q}$ is a reductive group. Suppose also that $E/\mathbb{Q}$ is a sufficiently large Galois extension that

- E splits G;
- E is totally imaginary;
- $B(\mathbb{Q},G)_{\{\infty\},\mathrm{basic}}$ is contained in the image of $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}),G(E))_{\mathrm{basic}}$.

In this case we will say that E is *acceptable* for G. The existence of some such E follows from the results recalled in section 2.3.

The results asserted in the rest of this section are all immediate consequences of the results recalled in section 2.3 and the results of 2.4.

Suppose moreover that Y is a compactifying $G(\mathbb{R})$-conjugacy class of cocharacters of G defined over $\mathbb{C}$, and that $\tau \in \mathrm{Aut}\,(\mathbb{C})$. Then there is a unique class $\boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}),G(E))_{\mathrm{basic}}$ such that

- $\kappa_G(\boldsymbol{\phi}_{G,Y,\tau}) = {}^{\rho^{-1}}\lambda_G(Y)\otimes(v(\rho)-v(\tau\rho))$; where $\rho : E \hookrightarrow \mathbb{C}$, and $v(\rho)$ denotes the corresponding infinite place of E (this is independent of the choice of ρ), and ${}^{\rho^{-1}}\lambda_G(Y)$ denotes the unique element of $\Lambda_G(E)$ mapping to $\lambda_G(Y)$ under ρ;
- and $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = \widehat{\boldsymbol{\lambda}}_G(Y - {}^{\tau}[Y]_G)$, where $[Y]_G$ is the unique G-conjugacy class of cocharacters containing Y.

We see that $\mathrm{res}^{\infty}\mathrm{loc}\boldsymbol{\phi}_{G,Y,\tau} = 1$.

If $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$, then $Y({}^{\tau}[Y]_G)_{^{\phi}G}$ is a compactifying conjugacy class of cocharacters of ${}^{\phi}G$ over $\mathbb{C}$, which we will denote ${}^{\tau,\phi}Y$. Note that if $G = T$ is a torus then $({}^{\phi}T, {}^{\tau,\phi}\{\mu\}) = (T,\{{}^{\tau}\mu\})$.

When $G_i/\mathbb{Q}$ are reductive groups and Y_i are a compactifying $G_i(\mathbb{R})$-conjugacy class of cocharacters of G_i over $\mathbb{C}$, we will write $f : (G_1,Y_1) \to (G_2,Y_2)$ to mean that $f : G_1 \to G_2$ is a morphism of algebraic groups over $\mathbb{Q}$ with $fY_1 \subset Y_2$. In this case, $f \circ \boldsymbol{\phi}_{G_1,Y_1,\tau} = \boldsymbol{\phi}_{G_2,Y_2,\tau}$ and, if $\phi \in \boldsymbol{\phi}_{G_1,Y_1,\tau}$, then

$$f : ({}^{\phi}G_1, {}^{\tau,\phi}Y_1) \to ({}^{f\circ\phi}G_2, {}^{\tau,f\circ\phi}Y_2).$$

If $G/\mathbb{Q}$ is reductive and if Y is a compactifying $G(\mathbb{R})$-conjugacy class of cocharacters of G over $\mathbb{C}$, we will write $\mathrm{Conj}\,_{E,\mathfrak{a}}(G,Y)$ for the set of triples (τ,ϕ,b), where

- $\tau \in \mathrm{Aut}\,(\mathbb{C})$;
- $\phi \in \boldsymbol{\phi}_{G,Y,\tau}$;
- $b \in G(\mathbb{A}_E^{\infty})$ satisfies $\mathrm{res}^{\infty}\mathrm{loc}_{\mathfrak{a}}\phi = {}^{b}1$.

We will call this the set of *conjugation data* for (G,Y). We will sometimes write ${}^{(\tau,\phi,b)}(G,Y) = ({}^{\phi}G, {}^{\tau,\phi}Y)$. We have

$$\mathrm{conj}_b : G \times \mathbb{A}^{\infty} \xrightarrow{\sim} {}^{\phi}G \times \mathbb{A}^{\infty}.$$

If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_E \in \mathcal{H}(E/\mathbb{Q})$ and $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and if $t \in T_{2,E}(\mathbb{A}_D)$ with ${}^{t}\inf_{E/\mathbb{Q}}\mathfrak{a}_E = \eta_{D/E,*}\mathfrak{a}_D$, and if $(\tau,\phi,b) \in \mathrm{Conj}\,_{E,\mathfrak{a}_E}(G,Y)$, then

$$\inf_{D/E,t}(\tau,\phi,b)=(\tau,\inf_{D/E,t}\phi,\boldsymbol{\nu}_\phi(t)b)\in \mathrm{Conj}_{D,\mathfrak{a}_D}(G,Y).$$

If $(\tau_1,\phi_1,b_1)\in \mathrm{Conj}_{E,\mathfrak{a}}(G,Y)$ and $(\tau_2,\phi_2,b_2)\in \mathrm{Conj}_{E,\mathfrak{a}}{}^{\tau_1,\phi_1,b_1}(G,Y)$, then

$$(\tau_2\tau_1,\phi_2\phi_1,b_2b_1)\in \mathrm{Conj}_{E,\mathfrak{a}}(G,Y),$$

and we have

$$^{(\tau_2,\phi_2,b_2)}\big({}^{(\tau_1,\phi_1,b_1)}(G,Y)\big)={}^{(\tau_2\tau_1,\phi_2\phi_1,b_2b_1)}(G,Y).$$

If $(\tau,\phi,b)\in \mathrm{Conj}_{E,\mathfrak{a}}(G_1,Y_1)$ and $f:(G_1,Y_1)\to(G_2,Y_2)$ then $f(\tau,\phi,b)=(\tau,f\circ\phi,f(b))\in \mathrm{Conj}_{E,\mathfrak{a}}(G_2,Y_2)$ and f induces a map

$$^{(\tau,\phi,b)}f:{}^{(\tau,\phi,b)}(G_1,Y_1)\longrightarrow {}^{f(\tau,\phi,b)}(G_2,Y_2).$$

Moreover

$$\mathrm{conj}_{f(b)}\circ f={}^{(\tau,\phi,b)}f\circ\mathrm{conj}_b.$$

If we fix $\tau\in\mathrm{Aut}(\mathbb{C})$ we will write $\mathrm{Conj}_{E,\mathfrak{a}}(G,Y)_\tau$ for the subset of $\mathrm{Conj}_{E,\mathfrak{a}}(G,Y)$ consisting of those triples with first element τ. The group $G(E)\times G(\mathbb{A}^\infty)$ acts transitively on $\mathrm{Conj}_{E,\mathfrak{a}}(G,Y)_\tau$ via

$$(\gamma,h)(\tau,\phi,b)=(\tau,{}^\gamma\phi,\gamma bh^{-1}).$$

The stabilizer of (τ,ϕ,b) is identified with ${}^\phi G(\mathbb{Q})$ via $\delta\mapsto(\delta,b^{-1}\delta b)$. We have

$$\inf_{D/E,t}((\gamma,h)(\tau,\phi,b))=(\gamma,h)\inf_{D/E,t}(\tau,\phi,b).$$

2.6. **Rigidification.** In this section we recall some additional structures in Kottwitz cohomology, which can be found in sections 2 and 9 of [ST]. But first we must recall the Serre torus and the Taniyama group.

Suppose that $E/\mathbb{Q}$ is a finite Galois extension. There is a torus $R_{E,\mathbb{C}}/\mathbb{Q}$ split by E with a cocharacter $\mu^{\mathrm{can}}=\mu_E^{\mathrm{can}}\in X_*(R_{E,\mathbb{C}})$ with the following universal property: if $T/\mathbb{Q}$ is any torus split by E and if $\mu\in X_*(T)(\mathbb{C})$, then there is a unique morphism $\widetilde{\mu}:R_{E,\mathbb{C}}\to T$ over $\mathbb{Q}$ such that $\mu=\widetilde{\mu}\circ\mu^{\mathrm{can}}$. Then we get a map

$$\begin{array}{ccc}\mathrm{Aut}(\mathbb{C}) & \longrightarrow & \mathrm{Aut}(R_{E,\mathbb{C}}/\mathbb{Q})\\ \tau & \longmapsto & [\tau]\end{array}$$

characterized by $\mu^{\mathrm{can}}=[\tau]\circ{}^\tau\mu^{\mathrm{can}}$. (The torus $R_{E,\mathbb{C}}$ is isomorphic to the restriction of scalars from $E\cap\mathbb{C}$ to $\mathbb{Q}$ of $\mathbb{G}_m$.) If $D\supset E$ is another finite Galois extension of $\mathbb{Q}$, then there is a natural map $N_{D/E}:R_{D,\mathbb{C}}\to R_{E,\mathbb{C}}$ over $\mathbb{Q}$ characterized by $N_{D/E}\circ\mu_D^{\mathrm{can}}=\mu_E^{\mathrm{can}}$.

There is also an exact sequence of tori

$$(0)\longrightarrow R^1_{E,\mathbb{C}}\longrightarrow R_{E,\mathbb{C}}\longrightarrow S_{E,\mathbb{C}}\longrightarrow(0)$$

over $\mathbb{Q}$, where $S_{E,\mathbb{C}}$ has the following universal property: if $T/\mathbb{Q}$ is any torus split by E and if $\mu\in X_*(T)(\mathbb{C})$ satisfies

- ${}^{c\tau}\mu={}^{\tau c}\mu$ for all $\tau\in\mathrm{Aut}(\mathbb{C})$,

- and ${}^{1+c}\mu$ is defined over $\mathbb{Q}$;

then there is a unique morphism $\widetilde{\mu} : S_{E,\mathbb{C}} \to T$ over $\mathbb{Q}$ such that $\mu = \widetilde{\mu} \circ \mu^{\mathrm{can}}$. The action of $\mathrm{Aut}\,(\mathbb{C})$ preserves the above short exact sequence; and $N_{D/E}$ takes $R^1_{D,\mathbb{C}}$ to $R^1_{E,\mathbb{C}}$, and $S_{D,\mathbb{C}}$ to $S_{E,\mathbb{C}}$. The torus $S_{E,\mathbb{C}}$ is usually referred to as the *Serre torus*. We have that $S_{E,\mathbb{C}}(\mathbb{Q})$ is a discrete subgroup of $S_{E,\mathbb{C}}(\mathbb{A}^\infty)$.

Lemma 2.3. *Suppose that* $\chi \in \mathbb{Z}[V_{E,\infty}]_0 \otimes X_*(S_{E,\mathbb{C}}) \subset X^*(T_{3,E}) \otimes X_*(S_{E,\mathbb{C}})$. *Then*

$$\prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} {}^\eta\chi = 1.$$

Langlands defines a canonical pro-algebraic group $\widetilde{S}_{E,\mathbb{C}}$ over $\mathbb{Q}$ (called the *Taniyama group*), which is an extension

$$(0) \longrightarrow S_{E,\mathbb{C}} \longrightarrow \widetilde{S}_{E,\mathbb{C}} \longrightarrow \mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q}) \longrightarrow (0),$$

together with a section

$$\mathrm{sp} : \mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q}) \longrightarrow \widetilde{S}_{E,\mathbb{C}}(\mathbb{A}^\infty);$$

such that the action of $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$ on $S_{E,\mathbb{C}}$ is via $\tau \mapsto [\tau]$. (Langlands actually defined an extension of $\mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ by $S_{E,\mathbb{C}}$. We have chosen to work with the pull back of that extension to $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$.) If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$ there is a natural map

$$\widetilde{N}_{D/E} : \widetilde{S}_{D,\mathbb{C}} \longrightarrow \widetilde{S}_{E,\mathbb{C}}$$

compatible with $N_{D/E} : S_{D,\mathbb{C}} \longrightarrow S_{E,\mathbb{C}}$, and satisfying $\widetilde{N}_{D/E} \circ \mathrm{sp}_D = \mathrm{sp}_E$.

Now we return to Kottwitz cohomology. If $E/\mathbb{Q}$ is a finite Galois extension there is a set $\mathcal{H}(E/\mathbb{Q})^+$ with a transitive action of $T_{2,E}(\mathbb{A}_E)$ and an equivariant surjection

$$\mathcal{H}(E/\mathbb{Q})^+ \twoheadrightarrow \mathcal{H}(E/\mathbb{Q}).$$

If $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$ we will write $\mathfrak{a}$ for its image in $\mathcal{H}(E/\mathbb{Q})$. The stabilizer of any element of $\mathcal{H}(E/\mathbb{Q})^+$ is

$$\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0 \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0 T_{2,\mathbb{Q}}(\mathbb{A}) \prod_{w \in V_E} \overline{E^\times (E^\times_\infty)^0}.$$

If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}^+_E \in \mathcal{H}(E/\mathbb{Q})^+$ and $\mathfrak{a}^+_D \in \mathcal{H}(D/\mathbb{Q})^+$, then we can find $t \in T_{2,E}(\mathbb{A}_D)$ with

$$\eta_{D/E,*}\mathfrak{a}^+_D = {}^t \inf_{D/E} \mathfrak{a}^+_E$$

in the notation of [ST]. The element of t is unique up to multiplication by an element of

$$\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^0_D \mathcal{E}^{\mathrm{glob}}(E/\mathbb{Q})^0_D T_{2,\mathbb{Q}}(\mathbb{A}) \prod_{w \in V_E} \overline{D^\times (D^\times_\infty)^0}.$$

Again in the notation of [ST]. If $C \supset D$ is a third finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_C^+ \in \mathcal{H}(C/\mathbb{Q})^+$ and if $t' \in T_{2,D}(\mathbb{A}_C)$ with $\eta_{C/D,*}\mathfrak{a}_C^+ = {}^{t'} \inf_{C/D} \mathfrak{a}_D^+$, then

$$\eta_{C/E,*}\mathfrak{a}_C^+ = {}^{t\eta_{D/E}(t')} \inf_{C/E} \mathfrak{a}_E^+.$$

Suppose that v is a place of $\mathbb{Q}$, that $\tau \in \mathrm{Aut}\,(\overline{\mathbb{Q}_v})$, that $T/\mathbb{Q}$ is a torus split by E and that $\mu \in X_*(T)(\overline{\mathbb{Q}_v})$. Then there is an element

$$\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} \in T(\mathbb{A}_E)/T(E)T(E_v)\overline{T(\mathbb{Q})T(\mathbb{R})^+}$$

with the following properties:

(1) $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau_1\tau_2} = \overline{b}_{\mathfrak{a}^+,v,{}^{\tau_2}\mu,\tau_1}\overline{b}_{\mathfrak{a}^+,v,\mu,\tau_2}$.

(2) If τ is continuous, then $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} = 1$.

(3) If $\rho : E^{\mathrm{ab}} \hookrightarrow \overline{\mathbb{Q}_v}$, if τ fixes the image of E in $\overline{\mathbb{Q}_v}$, and if $a_\rho \in \mathbb{A}_E^\times$ with $\rho \circ \mathrm{Art}\,_E(a_\rho) = \tau \circ \rho$, then

$$\overline{b}_{\mathfrak{a}^+,v,\mu,\tau} = \prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} \eta({}^{\rho^{-1}}\mu)(a_\rho)^{-1}.$$

In particular $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau}$ only depends on $\tau|_{E^{\mathrm{ab}} \cap \overline{\mathbb{Q}}_v}$.

(4) If $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ have the same image in $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$, then $\alpha^{-1}\mathrm{sp}(\tau|_{\mathbb{C}^{\mathrm{alg}}}) \in S_{E,\mathbb{C}}(\mathbb{A}_E^\infty)$ lifts $\overline{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\tau} \in S_{E,\mathbb{C}}(\mathbb{A}_E^\infty)/S_{E,\mathbb{C}}(E)$. Thus $\overline{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\tau}$ and $\alpha^{-1}\mathrm{sp}(\tau|_{\mathbb{C}^{\mathrm{alg}}})$ have a unique common lift

$$b_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha} \in R_{E,\mathbb{C}}(\mathbb{A}_E^\infty)/\overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}R^1_{E,\mathbb{C}}(E).$$

This element is independent of the choice of τ lifting the image of α in $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$.

(5) If $\chi : T \to T'$ over $\mathbb{Q}$, then $\overline{b}_{\mathfrak{a}^+,v,\chi\circ\mu,\tau} = \chi(\overline{b}_{\mathfrak{a}^+,v,\mu,\tau})$.

(6) If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$ and $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$, and if $t \in T_{2,E}(\mathbb{A}_D)$ with ${}^t \inf_{D/E} \mathfrak{a}_E^+ = \eta_{D/E,*}\mathfrak{a}_D^+$, then

$$\begin{aligned} \overline{b}_{\mathfrak{a}_E^+,v,\mu,\tau} &= \overline{b}_{\mathfrak{a}_D^+,v,\mu,\tau} \textstyle\prod_\rho ({}^{\rho^{-1}}\mu)(t_{w(\tau\rho)}/t_{w(\rho)})) \\ &= \overline{b}_{\mathfrak{a}_D^+,v,\mu,\tau} \textstyle\prod_\rho ({}^{\rho^{-1}}({}^\tau\mu/\mu)(t_{w(\rho)})), \end{aligned}$$

where ρ runs over embeddings $E \hookrightarrow \overline{\mathbb{Q}_v}$.

In particular, item 4 tells us that the elements $\overline{b}_{\mathfrak{a}^+,v,\mu,\tau}$ are closely connected to, and generalize, the cocycles that define the Taniyama group.

If $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ is a lift of $b_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ to $R_{E,\mathbb{C}}(\mathbb{A}_E^\infty)$, then there is a unique element $\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha} \in \boldsymbol{\phi}_{R_{E,\mathbb{C}},\{\mu^{\mathrm{can}}\},\tau}$ such that

$$(\tau, \widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}, \widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}) \in \mathrm{Conj}\,_{E,\mathfrak{a}}(R_{E,\mathbb{C}}, \{\mu^{\mathrm{can}}\}).$$

If we replace $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ by $h\gamma\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ with $h \in \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$ and $\gamma \in R^1_{E,\mathbb{C}}(E)$, then $\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ changes to ${}^\gamma\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$.

We have the following:

(1) If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ and $\gamma \in S_{E,\mathbb{C}}(E)$ then we can find $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ lifting γ such that we may take $\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\gamma\alpha} = {}^{\widetilde{\gamma}^{-1}}\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ and $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\gamma\alpha} = \widetilde{\gamma}^{-1}\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$.

(2)

$$\boldsymbol{\nu}_{\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}} = \prod_{\rho:E\hookrightarrow\mathbb{C}} {}^{\rho^{-1}}\mu^{\mathrm{can}}\circ(\pi_{w(\rho)}/\pi_{w(\tau\rho)}) = \prod_{\rho:E\hookrightarrow\mathbb{C}} {}^{\rho^{-1}}(\mu^{\mathrm{can}}/{}^{\tau}\mu^{\mathrm{can}})\circ\pi_{w(\rho)} \in X_*(R^1_{E,\mathbb{C}})(\mathbb{Q}),$$

where $\tau \in \mathrm{Aut}\,(\mathbb{C})$ has the same image in $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$ as α.

(3) Given $\alpha_i \in \widetilde{S}_{E,\mathbb{C}}(E)$ for $i = 1, 2$, there exists $\beta \in R^1_{E,\mathbb{C}}(E)$ such that

$$\beta\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha_1\alpha_2} \equiv [\overline{\alpha}_2^{-1}](\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha_1})\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha} \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$$

and

$${}^{\beta}\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha_1\alpha_2} = [\overline{\alpha}_2^{-1}](\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha_1})\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha_2},$$

where $\overline{\alpha}_2 \in \mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$ denotes the image of α_2.

(4) If $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ and $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$, then

$$\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}{}^{\sigma}(\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha})^{-1} \in R^1_{E,\mathbb{C}}(\mathbb{A}_E^\infty)R_{E,\mathbb{C}}(E) \subset R_{E,\mathbb{C}}(\mathbb{A}_E^\infty).$$

(5) Suppose that $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, that $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$, that $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D^+ = {}^t\inf_{D/E}\mathfrak{a}_E^+$. Suppose also that $\alpha_D \in \widetilde{S}_{D,\mathbb{C}}(D)$ and $\alpha_E \in \widetilde{S}_{E,\mathbb{C}}(E)$ have the same image in $\mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$, so that $\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D) \in S_{E,\mathbb{C}}(D)$. Choose $\widetilde{b}_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E}$ lifting $b_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E}$ and $\widetilde{b}_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D}$ lifting $b_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D}$. Then there exists $\beta \in R^1_{E,\mathbb{C}}(D)$ with

$$\widetilde{b}_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E} \equiv \beta(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))N_{D/E}(\widetilde{b}_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D}) \prod_{\rho:E\hookrightarrow\mathbb{C}} {}^{\rho^{-1}}({}^{\overline{\alpha}_E}\mu_E^{\mathrm{can}}/\mu_E^{\mathrm{can}})(t_{w(\rho)}) \bmod \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$$

and

$$\inf_{D/E,t}\widetilde{\phi}_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E} = {}^{\beta(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))}N_{D/E}\circ\widetilde{\phi}_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D} \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(D/\mathbb{Q})_{\mathfrak{a}_D^+}, R_{E,\mathbb{C}}(D)).$$

(Here $\overline{\alpha}_E$ denotes the image of α_E in $\mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$.)

Suppose that E is totally imaginary. Choose an embedding $\rho_0 : E \hookrightarrow \mathbb{C}$ and a set of representatives $H_\infty \ni 1$ for $\mathrm{Gal}\,(E/\mathbb{Q})/\mathrm{Gal}\,(E_{w(\rho_0)}/\mathbb{R})$. Recall that there is the global Weil group $W_{E^{\mathrm{ab}}/\mathbb{Q}}$, which fits into an extension

$$(0) \longrightarrow \mathbb{A}_E^\times/E^\times \longrightarrow W_{E^{\mathrm{ab}}/\mathbb{Q}} \longrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \longrightarrow (0),$$

together with a map $\varphi : W_{E^{\mathrm{ab}}/\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/\mathbb{Q})$ and a map

$$\theta_{\rho_0} : W_{E_{w(\rho_0)}/\mathbb{R}} = \langle E^\times_{w(\rho_0)}, j_{w(\rho_0)} :\ j^2_{w(\rho_0)} = -1 \text{ and } j_{w(\rho_0)}zj^{-1}_{w(\rho_0)} = {}^{c_{w(\rho_0)}}z\rangle \longrightarrow W_{E^{\mathrm{ab}}/\mathbb{Q}}.$$

All this is well defined up to conjugation by an element of $\overline{E^\times E_\infty^\times}/E^\times$. Also choose a section $s : \mathrm{Gal}\,(E/\mathbb{Q}) \to W_{E^{\mathrm{ab}}/\mathbb{Q}}$ such that

- $s(1) = 1$,
- $s(c_{w(\rho_0)}) = \theta_{\rho_0}(j_{w(\rho_0)})$,
- and, if $\eta \in H_\infty$, then $s(\eta c_{w(\rho_0)}) = s(\eta)s(c_{w(\rho_0)})$,

We may choose $\mathfrak{a}_0^+ \in \mathcal{H}(E/\mathbb{Q})^+$ compatible with these choices (as explained in section 6.4 of [ST]), and then we have

$$\overline{b}_{\mathfrak{a}_0^+,\infty,\mu,\tau} = \prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} \eta({}^{\rho_0^{-1}}\mu)(\widetilde{\tau^{\rho_0}}^{-1} s(\eta(\tau^{\rho_0})^{-1})^{-1} s(\eta)) \in T(\mathbb{A}_E)/T(E)T(E_\infty)\overline{T(\mathbb{Q})T(\mathbb{R})^0},$$

where $\widetilde{\tau^{\rho_0}}$ denotes any lift of $\tau^{\rho_0}|_{E^{\mathrm{ab}}}$ to $W_{E^{\mathrm{ab}}/\mathbb{Q}}$. Moreover, if $\tau \in \mathrm{Aut}\,(\mathbb{C})$, then there is an element $\alpha_0(\tau) \in \widetilde{S}_{E,\mathbb{C}}(E)$ above $\tau|_{\mathbb{C}^{\mathrm{alg}}}$ such that we may take

$$\widetilde{b}_{\mathfrak{a}_0^+,\infty,\mu^{\mathrm{can}},\alpha_0(\tau)} = \prod_{\eta \in \mathrm{Gal}\,(E/\mathbb{Q})} \eta({}^{\rho_0^{-1}}\mu)(\widetilde{\tau^{\rho_0}}^{-1} s(\eta(\tau^{\rho_0})^{-1})^{-1} s(\eta)) \in T(\mathbb{A}_E^\infty).$$

3. Deligne's Shimura varieties

3.1. **Deligne's Shimura data.** In Deligne's formalism, Shimura varieties are attached to 'Shimura data'. In this section we will recall Deligne's definition of 'Shimura data'.

By a *(Deligne) Shimura datum* we shall mean a pair (G, Y), where $G/\mathbb{Q}$ is a reductive group and $Y \subset X_*(G)(\mathbb{C})$ is a compactifying $G(\mathbb{R})$-conjugacy class of miniscule cocharacters. The smooth manifold Y has a unique structure of a complex manifold such that $\sqrt{-1}$ acts on $T_\mu Y = \mathrm{Lie}\, G(\mathbb{R})/\mathrm{Lie}\,\mathrm{Stab}_{G(\mathbb{R})}(\mu)$ by $\mathrm{ad}\,\mu(\sqrt{-1})$. This will be explained below. Moreover if $\mu \in Y$ there is a unique parabolic subgroup $P_\mu^- \subset G$ over $\mathbb{C}$ such that $\mathrm{Lie}\, P_\mu^-$ is the sum of the weight 0 and -1 spaces in $\mathrm{Lie}\, G$ for $\mathrm{ad}\,\mu$. Then $P_\mu^-(\mathbb{C}) \cap G(\mathbb{R}) = \mathrm{Stab}_{G(\mathbb{R})}(\mu)$ and so there is a well defined map

$$\begin{array}{rcl} Y & \longrightarrow & G(\mathbb{C})/P_\mu^-(\mathbb{C}) \\ \mathrm{conj}_h \circ \mu & \longmapsto & hP_\mu^-(\mathbb{C}). \end{array}$$

This is a diffeomorphism onto an open subset of $G(\mathbb{C})/P_\mu^-(\mathbb{C})$.

By a morphism $\phi : (G_1, Y_1) \to (G_2, Y_2)$ we mean morphism $\phi : G_1 \to G_2$ of algebraic groups over $\mathbb{Q}$ such that $\phi(Y_1) \subset Y_2$. For instance if $\gamma \in G^{\mathrm{ad}}(\mathbb{Q})_{\mathbb{R}}$, then

$$\mathrm{conj}_\gamma : (G, Y) \longrightarrow (G, Y).$$

We will write

$$E(G, Y) = \mathbb{C}^{\mathrm{Stab}_{\mathrm{Aut}\,(\mathbb{C})}([Y]_{G(\mathbb{C})})} \subset \mathbb{C}$$

for the field of definition of the $G(\mathbb{C})$ conjugacy class $[Y]_{G(\mathbb{C})}$ containing Y. It is a number field called the *reflex field of* (G, Y). It comes with a preferred embedding

$$\imath_{(G,Y)} : E(G, Y) \hookrightarrow \mathbb{C}.$$

The variety $[Y]_G$ can be defined over $E(G, Y)$.

This is not how a Shimura datum is usually defined, but is easily seen to be equivalent to it, as we now explain. Write $\mathbb{S}$ for the restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of $\mathbb{G}_m$ and identify

$$\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$$

so that $z \otimes w \in \mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times$ corresponds to $(zw, \overline{z}w)$. There is a natural inclusion $\mathbb{G}_m \hookrightarrow \mathbb{S}$. Deligne defined a Shimura datum to be a pair (G, X), where $G/\mathbb{Q}$ is a reductive algebraic group and X is a $G(\mathbb{R})$-conjugacy class of morphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$ satisfying the following axioms:

(1) if $h \in X$ then the adjoint action of $\mathbb{S} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$ on $(\mathrm{Lie}\, G)_{\mathbb{C}}$ has all its characters in the set $\{(1, -1), (0, 0), (-1, 1)\}$;
(2) if $h \in X$ then $\mathrm{ad}\, h(i)$ is a Cartan involution for the adjoint group G^{ad}.

If $h \in X$ then $h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m^2 \to G_{\mathbb{C}}$ has the form $(\mu_h, {}^c\mu_h)$ for a unique cocharacter $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$. We define

$$Y_X = \{\mu_h : \ h \in X\}.$$

It is easily seen to be a basic $G(\mathbb{R})$-conjugacy class of miniscule cocharacters of G. (Note that $h|_{\mathbb{G}_m} = \mu_h{}^c\mu_h$, and that if $h \in X$, then $\mathrm{ad}\, h|_{\mathbb{G}_m} = 1$.) It is moreover compactifying because $\mathrm{ad}\, h(i) = \mathrm{ad}\, \mu_h(i)^c\mu_h(-i) = \mathrm{ad}\, \mu_h(-1)\mathrm{ad}\, (\mu_h{}^c\mu_h)(-i) = \mathrm{ad}\, \mu_h(-1)$. Conversely if (G, Y) is a Shimura datum in our sense and if $\mu \in Y$, then μ and μ^c commute (as μ^c is a central character times μ^{-1}) and so

$$(\mu, {}^c\mu) : \mathbb{G}_m^2 \longrightarrow G_{/\mathbb{C}}$$

descends to a homomorphism

$$h_\mu : \mathbb{S} \longrightarrow G_{/\mathbb{R}}.$$

Note that $\mathrm{ad}\, h_\mu(i) = \mathrm{ad}\, (\mu(i)({}^c\mu)(-i)) = \mathrm{ad}\, \mu(-1)\mathrm{ad}\, (\mu^c\mu)(-i) = \mathrm{ad}\, \mu(-1)$. Thus

$$(G, \{h_\mu : \mu \in Y\})$$

is a Shimura datum in Deligne's sense. These two constructions are easily seem to be inverse to one another. By definition, $E(G, Y_X)$ coincides with the usual reflex field of (G, X).

Write $\mathbb{S}^1$ for the kernel of the norm map $\mathbb{S} \to \mathbb{G}_m$. Then there is an identification $\mathbb{S}/\mathbb{G}_m \xrightarrow{\sim} \mathbb{S}^1$ by the map which on $\mathbb{R}$-points sends $z \in \mathbb{C}^\times$ to $z/{}^cz$. If $h \in X$, then $\mathrm{ad}\, h$ factors through a unique map $u_h : \mathbb{S}^1 \to G^{\mathrm{ad}}$. We have

$$u_h|_{\mathbb{S}^1(\mathbb{R})} = \mathrm{ad}\, \mu_h|_{(\mathbb{C}^\times)^{N_{\mathbb{C}/\mathbb{R}}=1}}.$$

According to proposition 5.9 and theorem 2.14 of [Mi3], the complex structure on $T_hX = \mathrm{Lie}\, G(\mathbb{R})/\mathrm{Lie}\,\mathrm{Stab}_{G(\mathbb{R})}(h)$ is such that $(\mathbb{C}^\times)^{N_{\mathbb{C}/\mathbb{R}}=1}$ acts by the adjoint action of u_h, i.e. by $\mathrm{ad}\, \mu_h$. This explains the assertion in the second paragraph of this section.

In [D2], Deligne imposed a further condition on his Shimura data (G, X):

(3) G^{ad} has no $\mathbb{Q}$ factors on which the projection of any (and hence every) $h \in X$ is trivial, or what amounts to the same thing: the projection of μ_h is trivial.

Most subsequent authors have continued to impose this assumption. In this paper we will not impose this condition on a Shimura datum. If a Shimura datum (G, Y) does satisfy the additional condition that G^{ad} has no $\mathbb{Q}$ factor on which the projection of any μ is trivial, we will call (G, Y) a *NCF-Shimura datum.* (Here 'NCF' stands for 'no compact factor'.)

An element $\mu \in Y$ is called *special* if it factors through a sub-torus $T \subset G$ which is defined over $\mathbb{Q}$. We will call it *E-special* if we may choose T such that in addition T is split by E.

Lemma 3.1. *(1) If $\mu \in Y$ is special it factors through a maximal torus defined over $\mathbb{Q}$.*

(2) If $\mu \in Y$ is special and factors through a torus T defined over $\mathbb{Q}$, then $T^{\mathrm{ad}}(\mathbb{R})$ is compact, i.e. c acts on $X_*(T^{\mathrm{ad}})$ by -1.

(3) If $T \subset G$ is a maximal torus defined over $\mathbb{Q}$ and if $T^{\mathrm{ad}}(\mathbb{R})$ is compact then there is a $\mu \in Y$ which factors through T.

(4) *If G contains a maximal torus T defined over $\mathbb{Q}$ and split by E with $T^{\mathrm{ad}}(\mathbb{R})$ compact, then the E-special points in Y are dense. In any case the special points in Y are dense.*
(5) *If $\mu \in Y$ is special and $E/\mathbb{Q}$ is Galois, then μ is E-special if and only if μ is defined over E.*
(6) *If $E/\mathbb{Q}$ is finite Galois and if $\mu \in Y$ is E-special factoring through a torus $T \subset G$ defined over $\mathbb{Q}$ and split by E, then there is a commutative diagram*

$$\begin{array}{ccccc} R^1_{E,\mathbb{C}} & \longrightarrow & Z(G) \cap T & \subset & Z(G) \\ \downarrow & & \downarrow & & \downarrow \\ R_{E,\mathbb{C}} & \stackrel{\widetilde{\mu}}{\longrightarrow} & T & \subset & G \\ \downarrow & & \downarrow & & \downarrow \\ S_{E,\mathbb{C}} & \stackrel{\mathrm{ad}\,\widetilde{\mu}}{\longrightarrow} & T^{\mathrm{ad}} & \subset & G^{\mathrm{ad}}. \end{array}$$

Moreover the the restriction $\widetilde{\mu}|_{R^1_{E,\mathbb{C}}}$ does not depend on μ or T. We will denote it $\widetilde{\mu}_{Y,E}$.

Proof: For the first part suppose that μ factors through a torus $T \subset G$ defined over $\mathbb{Q}$. Then one can replace T by a maximal torus of $Z_G(T)$ defined over $\mathbb{Q}$.

For the second part note that T^{ad} embeds over $\mathbb{R}$ into the inner form of G^{ad} determined by the cocycle $c \mapsto \mathrm{ad}\,\mu(-1)$, whose real points are compact.

For the third part choose any $\mu_1 \in Y$ and chose a maximal torus $T_1 \subset G$ defined over $\mathbb{R}$ through which μ_1 factors. Then $T_1^{\mathrm{ad}}(\mathbb{R})$ is compact (as in part 2)). Thus T and T_1 are fundamental tori in $G_{/\mathbb{R}}$ and hence $T = gT_1g^{-1}$ for some $g \in G(\mathbb{R})$. Then $\mu = g\mu_1 g^{-1}$ will do.

For the first assertion of the fourth part, because $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$, it suffices to see that there is some E-special point. This follows from the previous part. For the second assertion choose a maximal torus $T_1 \subset G$ defined over $\mathbb{R}$ with $T_1^{\mathrm{ad}}(\mathbb{R})$ compact. Then T_1 is $G(\mathbb{R})$-conjugate to some maximal torus $T \subset G$ defined over $\mathbb{Q}$, and we see that $T^{\mathrm{ad}}(\mathbb{R})$ is also compact. Choosing a finite Galois extension $E/\mathbb{Q}$ which splits T, and the second assertion follows from the first. (The facts about algebraic groups used in this paragraph were recalled in section 2.2.)

For the fifth part note that if μ factors through a torus $T \subset G$ defined over $\mathbb{Q}$ and split over E, then μ, like any cocharacter of T, is defined over E. Conversely, if μ is defined over E and factors through a torus $T_1 \subset G$ defined over $\mathbb{Q}$, then let T be the minimal subtorus of T_1 defined over $\mathbb{Q}$ through which μ factors. Because $X_*(T_1)^{\mathrm{Gal}(\overline{E}/E)}$ is $\mathrm{Gal}(\overline{E}/\mathbb{Q})$-invariant, we see that T splits over E.

For the sixth part note that one, and hence every, complex conjugation acts on $X_*(T^{\mathrm{ad}})$ by -1. If μ_1 and $\mu_2 \in Y$ are two E-special points, then the composites $\mu_i : \mathbb{G}_m \to G \to C(G)$ are equal and hence so are the composites $\widetilde{\mu}_i : R_{E,\mathbb{C}} \to G \to C(G)$. Because $Z(G) \to C(G)$ is an isogeny we see that $\widetilde{\mu}_1|_{R^1_{E,\mathbb{C}}} = \widetilde{\mu}_2|_{R^1_{E,\mathbb{C}}}$ □

3.2. **Langlands' theory of conjugation of Shimura data.** Let $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$. Also fix choices of $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ lifting $b_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ and hence also of $\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ as in section 2.6.

Suppose that $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ and $\tau \in \mathrm{Aut}\,(\mathbb{C})$ have the same image in $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$. Suppose also that $\mu \in Y$ is E-special, and choose a torus $T \subset G$ defined over $\mathbb{Q}$ and split by E, through which μ factors. Then $\widetilde{\mu} : R_{E,\mathbb{C}} \to T$ over $\mathbb{Q}$.

We define

$$\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = \widetilde{\mu}(\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}) \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$$

and

$$\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = \widetilde{\mu}(\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}) \in T(\mathbb{A}_E^\infty) \subset G(\mathbb{A}_E^\infty).$$

These depend on the the choice of $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$. If we change $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$ by $h\gamma$ with $h \in \overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$ and $\gamma \in R^1_{E,\mathbb{C}}(E)$, then $\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ changes to $\widetilde{\mu}_{Y,E}(h\gamma)\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ changes to ${}^{\widetilde{\mu}_{Y,E}(\gamma)}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$.

We have the following observations, which all follow from the corresponding results for $\widetilde{\phi}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$ and $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu_{\mathbb{C}}^{\mathrm{can}},\alpha}$:

(1) $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}} \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} = {}^{\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}} 1$.

(2) $\boldsymbol{\nu}_{\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}} = \prod_{\rho: E \hookrightarrow \mathbb{C}} ({}^{\rho^{-1}}(\mu/{}^\tau\mu)) \circ \pi_{v(\rho)}$, which by lemmas 2.3 is valued in $Z(G)$.

(3) $[\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}] = \boldsymbol{\phi}_{G,Y,\tau} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$. (This is seen by reduction to the case $G = R_{E,\mathbb{C}}$ and using the results listed in section 2.5.)

(4) $(\tau, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}) \in \mathrm{Conj}_{\,E,\mathfrak{a}}(G,Y)$.

(5) Given $\tau_1, \tau_2 \in \mathrm{Aut}\,(\mathbb{C})$ and $\alpha_i \in \widetilde{S}_{E,\mathbb{C}}(E)$ having the same image in $\mathrm{Gal}\,(\mathbb{C}^{\mathrm{alg}}/\mathbb{Q})$ as τ_i, there exists $\beta \in Z(G)(E)$, independent of $\mu \in Y$ E-special, such that

$$\beta \widetilde{b}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} \equiv \widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu} \widetilde{b}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} \bmod \overline{Z(G)(\mathbb{Q})}$$

and

$${}^{\beta}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu} \phi_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}.$$

(6) If $\gamma \in S_{E,\mathbb{C}}(E)$ then γ has a lift $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ (independent of μ) such that

$$\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu} = {}^{\widetilde{\mu}(\widetilde{\gamma})}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$$

and

$$\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu} = \widetilde{\mu}(\widetilde{\gamma})\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}.$$

(7) Suppose that $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, that $\mathfrak{a}_E^+ \in \mathcal{H}(E/\mathbb{Q})^+$, that $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D^+ = {}^t \inf_{D/E} \mathfrak{a}_E^+$. Suppose also that $\alpha_D \in \widetilde{S}_{D,\mathbb{C}}(D)$ and $\alpha_E \in \widetilde{S}_{E,\mathbb{C}}(E)$ have images $\tau_D \in \mathrm{Gal}\,(D^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ and $\tau_E \in \mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ with $\tau_E = \tau_D|_{E^{\mathrm{ab}}\cap\mathbb{C}}$. Thus $\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D) \in$

$S_{E,\mathbb{C}}(D)$. Choose $\widetilde{b}_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E}$ lifting $b_{\mathfrak{a}_E^+,\infty,\mu_E^{\mathrm{can}},\alpha_E}$ and $\widetilde{b}_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D}$ lifting $b_{\mathfrak{a}_D^+,\infty,\mu_D^{\mathrm{can}},\alpha_D}$. Then there exists $\beta \in Z(G)(D)$ with

$$\widetilde{b}_{E,\mathfrak{a}_E^+,\tau_E,\alpha_E,\mu}\boldsymbol{\nu}_{\widetilde{\phi}_{E,\mathfrak{a}_E^+,\tau_E,\alpha_E,\mu}}(t) \equiv \beta\widetilde{\mu}(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))\widetilde{b}_{D,\mathfrak{a}_D^+,\tau_D,\alpha_D,\mu} \bmod \overline{Z(G)(\mathbb{Q})}$$

and

$$\inf_{D/E,t} \widetilde{\phi}_{E,\mathfrak{a}_E^+,\tau_E,\alpha_E,\mu} = {}^{\beta\widetilde{\mu}(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))}\widetilde{\phi}_{D,\mathfrak{a}_D^+,\tau_D,\alpha_D,\mu} \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(D/\mathbb{Q})_{\mathfrak{a}_D^+}, G(D))_{\mathrm{basic}}.$$

We will write

$$\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu} = \mathrm{ad}\,\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), G^{\mathrm{ad}}\,(E)),$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha,\mu} = \mathrm{ad}\,\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \in G^{\mathrm{ad}}\,(\mathbb{A}_E^\infty).$$

As the notation suggests these do not depend on the choice of $\mathfrak{a}^+$ or $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$. (This follows from the properties listed above.) If $\gamma \in S_{E,\mathbb{C}}(E)$ then

$$\phi^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = {}^{(\mathrm{ad}\,\widetilde{\mu})(\gamma)}\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = (\mathrm{ad}\,\widetilde{\mu})(\gamma)b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}.$$

The cocycle $\phi^{\mathrm{ad}}_{E,\tau,\alpha_0(\tau),\mu}$ equals the cocycle $\sigma \mapsto c_\sigma(\tau,\mu_{\mathrm{ad}})^{-1}$ of section 6 of [L]. Moreover the element $b^{\mathrm{ad}}_{E,\tau,\alpha_0(\tau),\mu} \in G^{\mathrm{ad}}\,(\mathbb{A}_E^\infty)$ equals the element denoted $\mathrm{ad}\,\widetilde{b}(\tau,\mu)^{-1}$ in section 6 of [L]. (Recall that $\alpha_0(\tau)$ was defined at the end of section 2.6.) Note that Langlands does not mention the chosen lift α in his notation. This is presumably because, as we just pointed out, there is a canonical relationship between these quantities for different choices of α. Nonetheless we find it less confusing to keep track of the α.

Suppose that $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, and that $\alpha_D \in \widetilde{S}_{D,\mathbb{C}}(D)$ and $\alpha_E \in \widetilde{S}_{E,\mathbb{C}}(E)$ have images $\tau_D \in \mathrm{Gal}\,(D^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ and $\tau_E \in \mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ with $\tau_E = \tau_D|_{E^{\mathrm{ab}}\cap\mathbb{C}}$. Then $\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D) \in S_{E,\mathbb{C}}(D)$ and

$$b^{\mathrm{ad}}_{E,\tau_E,\alpha_E,\mu} = (\mathrm{ad}\,\widetilde{\mu})(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))b^{\mathrm{ad}}_{D,\tau_D,\alpha_D,\mu}$$

and

$$\inf_{D/E} \phi^{\mathrm{ad}}_{E,\tau_E,\alpha_E,\mu} = {}^{\widetilde{\mu}(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))}\phi^{\mathrm{ad}}_{D,\tau_D,\alpha_D,\mu} \in Z^1(\mathrm{Gal}\,(D/\mathbb{Q}), G^{\mathrm{ad}}\,(D)).$$

Following Langlands we will set

$${}^{\tau,\mu,\alpha}(G,Y) = ({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y) = {}^{(\tau,\phi_{E,\mathfrak{a}^+,\tau,\alpha,\mu},b_{E,\mathfrak{a}^+,\tau,\alpha,\mu})}(G,Y),$$

so that

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}} : G \times \mathbb{A}^\infty \xrightarrow{\sim} {}^{\tau,\mu,\alpha}G \times \mathbb{A}^\infty$$

and ${}^\tau\mu \in {}^{\tau,\mu,\alpha}Y$. Note that

$${}^{\tau,\mu,\alpha}(G,Y) = ({}^{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}G, {}^{\tau,\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}Y),$$

and so ${}^{\tau,\mu,\alpha}(G,Y)$ does not depend on the choice of $\mathfrak{a}^+$. This notation is consistent with Langlands notation in [L], except again Langlands suppresses the choice of α in his notation. If $\gamma \in S_{E,\mathbb{C}}(E)$, then there is a canonical identification

$$\mathrm{conj}_{(\mathrm{ad}\,\widetilde{\mu})(\gamma)} : {}^{\tau,\mu,\alpha}(G,Y) \xrightarrow{\sim} {}^{\tau,\mu,\alpha\gamma^{-1}}(G,Y)$$

and

$$b^{\mathrm{ad}}_{E,\tau,\alpha\gamma^{-1},\mu} = (\mathrm{ad}\,\widetilde{\mu})(\gamma) b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}.$$

This may be seen as explaining Langlands choice to suppress the α in his notation, but again we feel it is clearer to make it explicit.

Suppose that $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, and that $\alpha_D \in \widetilde{S}_{D,\mathbb{C}}(D)$ and $\alpha_E \in \widetilde{S}_{E,\mathbb{C}}(E)$ have images $\tau_D \in \mathrm{Gal}\,(D^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ and $\tau_E \in \mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ with $\tau_E = \tau_D|_{E^{\mathrm{ab}}\cap\mathbb{C}}$. Then $\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D) \in S_{E,\mathbb{C}}(D)$ and

$$\mathrm{conj}_{(\mathrm{ad}\,\widetilde{\mu})(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))} : {}^{\tau_D,\mu,\alpha_D}(G,Y) \xrightarrow{\sim} {}^{\tau_E,\mu,\alpha_E}(G,Y)$$

and

$$b^{\mathrm{ad}}_{E,\tau_E,\alpha_E,\mu} = (\mathrm{ad}\,\widetilde{\mu})(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D)) b^{\mathrm{ad}}_{D,\tau_D,\alpha_D,\mu}.$$

As

$$\phi^{\mathrm{ad}}_{E,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = \phi^{\mathrm{ad}}_{E,\tau_1,\alpha_1,{}^{\tau_2}\mu}\phi^{\mathrm{ad}}_{E,\tau_2,\alpha_2,\mu},$$

we see that

$${}^{\tau_1\tau_2,\mu,\alpha_1\alpha_2}(G,Y) = {}^{\tau_1,{}^{\tau_2}\mu,\alpha_1}({}^{\tau_2,\mu,\alpha_2}(G,Y)).$$

Similarly

$$b^{\mathrm{ad}}_{E,\tau_1\tau_2,\alpha_1\alpha_2,\mu} = b^{\mathrm{ad}}_{E,\tau_1,\alpha_1,{}^{\tau_2}\mu} b^{\mathrm{ad}}_{E,\tau_2,\alpha_2,\mu}.$$

If $f:(G_1,Y_1)\to(G_2,Y_2)$ is a morphism of Shimura data and $\mu_1 \in Y_1$ is special, then we get a morphism

$${}^{\tau,\mu_1,\alpha}f : {}^{\tau,\mu_1,\alpha}(G_1,Y_1) \longrightarrow {}^{\tau,f(\mu_1),\alpha}(G_2,Y_2).$$

Moreover

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f(\mu_1)}} \circ f = {}^{\tau,\mu_1,\alpha}f \circ \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}}.$$

If $\mu_1,\mu_2 \in Y$ are both E-special then we set

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2} = \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2}\widetilde{\phi}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_1} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G(E))$$

and

$$b_{E,\tau,\alpha,\mu_1,\mu_2} = \widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2}\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_1} \in G(\mathbb{A}_E^\infty).$$

As the notation suggests, these do not depend on the choices of $\mathfrak{a}^+$ and $\widetilde{b}_{\mathfrak{a}^+,\infty,\mu^{\mathrm{can}},\alpha}$.

(1)

$$\mathrm{ad}\,b_{E,\tau,\alpha,\mu_1,\mu_2} = b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}(b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1})^{-1} \in G^{\mathrm{ad}}(\mathbb{A}_E^\infty)$$

and

$$\phi_{E,\tau,\alpha,\mu_1,\mu_2} \longmapsto \phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}(\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1})^{-1} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G^{\mathrm{ad}}(E)).$$

(2)
$$\phi_{E,\tau,\alpha,\mu_1,\mu_2}(\sigma) = b_{E,\tau,\alpha,\mu_1,\mu_2}\mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}(\sigma)}({}^{\sigma}b_{E,\tau,\alpha,\mu_1,\mu_2})^{-1}.$$

(3) If $\widetilde{\gamma} \in R_{E,\mathbb{C}}(E)$ maps to $\gamma \in S_{E,\mathbb{C}}(E)$ then
$$b_{E,\tau,\alpha\gamma,\mu_1,\mu_2} = \widetilde{\mu}_2(\widetilde{\gamma})^{-1} b_{E,\tau,\alpha,\mu_1,\mu_2}\widetilde{\mu}_1(\widetilde{\gamma})$$
and
$$\phi_{E,\tau,\alpha\gamma,\mu_1,\mu_2} = \mathrm{conj}_{\widetilde{\mu}_1(\widetilde{\gamma})}({}^{\widetilde{\mu}_1(\widetilde{\gamma})^{-1}\widetilde{\mu}_2(\widetilde{\gamma})}\phi_{E,\tau,\alpha,\mu_1,\mu_2}).$$

(4) Suppose that $D \supset E$ are finite Galois extensions of $\mathbb{Q}$, and that $\alpha_D \in \widetilde{S}_{D,\mathbb{C}}(D)$ and $\alpha_E \in \widetilde{S}_{E,\mathbb{C}}(E)$ have images $\tau_D \in \mathrm{Gal}\,(D^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ and $\tau_E \in \mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q})$ with $\tau_E = \tau_D|_{E^{\mathrm{ab}}\cap\mathbb{C}}$. Then $\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D) \in S_{E,\mathbb{C}}(D)$ and
$$b_{E,\tau_E,\alpha_E,\mu_1,\mu_2} = \widetilde{\mu}_2(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))b_{D,\tau_D,\alpha_D,\mu_1,\mu_2}\widetilde{\mu}_1(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))^{-1}$$
and
$$\inf_{D/E}\phi_{E,\tau_E,\alpha_E,\mu_1,\mu_2} = \mathrm{conj}_{\widetilde{\mu}_1(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))}({}^{\widetilde{\mu}_1(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))^{-1}\widetilde{\mu}_2(\alpha_E^{-1}\widetilde{N}_{D/E}(\alpha_D))}\phi_{D,\tau_D,\alpha_D,\mu_1,\mu_2})$$

(5) $[\phi_{E,\tau,\alpha,\mu_1,\mu_2}] \in H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha}G)$ is trivial, so that
$$\phi_{E,\tau,\alpha,\mu_1,\mu_2}(\sigma) = \gamma_{E,\tau,\alpha,\mu_1,\mu_2}\mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu_1}(\sigma)}({}^{\sigma}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2})$$
for some $\gamma_{E,\tau,\alpha,\mu_1,\mu_2} \in G(E)$ well defined up to right multiplication by an element of ${}^{\tau,\mu_1,\alpha}G(\mathbb{Q})$. We see that
$$\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2} = {}^{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu_2}$$
and
$$\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}} : {}^{\tau,\mu_1,\alpha}G \xrightarrow{\sim} {}^{\tau,\mu_2,\alpha}G,$$
and
$$b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2} \in {}^{\tau,\mu_2,\alpha}G(\mathbb{A}^\infty).$$
Moreover
$$\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}}({}^{\tau,\mu_1,\alpha}Y) = {}^{\tau,\mu_2,\alpha}Y.$$

The cocycle $\phi_{E,\tau,\alpha_0(\tau),\mu_1,\mu_2} \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\tau,\mu_1,\alpha_0(\tau)}G)$ equals the cocycle denoted $\sigma \mapsto \gamma_\sigma$ in 'the first lemma of comparison' in section 6 of [L]. Moreover $b_{E,\tau,\alpha_0(\tau),\mu_1,\mu_2} \in G(\mathbb{A}_E^\infty)$ is the element denoted $B(\tau) = B(\tau,\mu_1,\mu_2)$ in section 6 of [L]. Finally the element $\gamma_{E,\tau,\alpha_0(\tau),\mu_1,\mu_2} \in G(E)$ is denoted u in the 'second lemma of comparison' in section 6 of [L]. (Again recall that $\alpha_0(\tau)$ was defined at the end of section 2.6.)

3.3. **Deligne's Shimura varieties.** If $U \subset G(\mathbb{A}^\infty)$ is an open compact subgroup, we will write U_v^{ad} for the image of U in $G^{\mathrm{ad}}(\mathbb{Q}_v)$ for any finite place v of $\mathbb{Q}$. We will call an open compact subgroup $U \subset G(\mathbb{A}^\infty)$ *sufficiently small* if for no integer $m > 1$ does $G(\mathbb{Q})^{\mathrm{ad}}$ and each U_v^{ad} contain an element of exact order m. Every open compact subgroup of $G(\mathbb{A}^\infty)$ has an open normal subgroup which is sufficiently small.

Given a Shimura datum (G, Y) and a sufficiently small compact open subgroup $U \subset G(\mathbb{A}^\infty)$ the complex analytic manifold

$$\mathrm{Sh}(G, Y)_U(\mathbb{C}) = G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y)$$

arises from a unique smooth quasi-projective variety $\mathrm{Sh}(G, Y)_U$ over $\mathbb{C}$. Moreover to each morphism $f : (G_1, Y_1) \to (G_2, Y_2)$ of Shimura data, each sufficiently small open compact subgroup $U_i \subset G_i(\mathbb{A}^\infty)$ and each $g \in G_2(\mathbb{A}^\infty)$ such that $gf(U_1)g^{-1} \subset U_2$ the map

$$\begin{array}{rcl} G_1(\mathbb{Q})\backslash(G_1(\mathbb{A}^\infty)/U_1 \times Y_1) & \longrightarrow & G_2(\mathbb{Q})\backslash(G_2(\mathbb{A}^\infty)/U_2 \times Y_2) \\ G_1(\mathbb{Q})(hU_1, x) & \longmapsto & G_2(\mathbb{Q})(f(h)g^{-1}U_2, f(x)), \end{array}$$

is holomorphic and arises from an algebraic map

$$\mathrm{Sh}(g, f) : \mathrm{Sh}(G_1, Y_1)_{U_1} \longrightarrow \mathrm{Sh}(G_2, Y_2)_{U_2}.$$

If $U \rhd V$ are sufficiently small open compact subgroups then

$$\mathrm{Sh}(1, 1) : \mathrm{Sh}(G, Y)_V/U \stackrel{\sim}{\longrightarrow} \mathrm{Sh}(G, Y)_U,$$

where $u \in U$ acts as $\mathrm{Sh}(u, 1)$. Thus for any open compact subgroup $U \subset G(\mathbb{A}^\infty)$ we can define a normal, quasi-projective variety over $\mathbb{C}$

$$\mathrm{Sh}(G, Y)_U = \mathrm{Sh}(G, Y)_V/U$$

for any sufficiently small, open, normal subgroup $V \lhd U$. (This is independent of the choice of V.) If $G = T$ is a torus then we have an isomorphism

$$\begin{array}{rcl} \Pi_{T,\{\mu\}} : T(\mathbb{Q})\backslash T(\mathbb{A}^\infty)/U & \stackrel{\sim}{\longrightarrow} & \mathrm{Sh}(T, \{\mu\})_U(\mathbb{C}) \\ T(\mathbb{Q})tU & \longmapsto & [(t, \mu)] \end{array}$$

Note that

(1) If $f_1 : (G_1, Y_1) \to (G_2, Y_2)$ and $f_2 : (G_2, Y_2) \to (G_3, Y_3)$ and if $U_i \subset G_i(\mathbb{A}^\infty)$ is a sufficiently small open compact subgroup and if $g_i \in G_i(\mathbb{A}^\infty)$ (for $i = 2, 3$) satisfy $g_2 f_1(U_1) g_2^{-1} \subset U_2$ and $g_3 f_2(U_2) g_3^{-1} \subset U_3$, then

$$\mathrm{Sh}(g_3, f_2) \circ \mathrm{Sh}(g_2, f_1) = \mathrm{Sh}(g_3 f_2(g_2), f_2 \circ f_1).$$

In particular as U varies over sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$ the filtered inverse system $\{\mathrm{Sh}(G, Y)_U\}$ (with transition maps $\mathrm{Sh}(1, 1)$) has a left action of $G(\mathbb{A}^\infty)$, where g acts by $\mathrm{Sh}(g, 1)$.

(2) The maps $\mathrm{Sh}(g, 1) : \mathrm{Sh}(G, Y)_{U_1} \to \mathrm{Sh}(G, Y)_{U_2}$ are finite and faithfully flat of degree $[U_2 Z(G)(\mathbb{Q}) : gU_1 g^{-1} Z(G)(\mathbb{Q})]$. If U_2 is sufficiently small then this map is etale.

(3) If $\gamma \in G(\mathbb{Q})$ and $u \in U$ then $\mathrm{Sh}(u\gamma^{-1}, \mathrm{conj}_\gamma)$ is the identity on $\mathrm{Sh}(G,Y)_U$. In particular if $z \in \overline{Z(G)(\mathbb{Q})}$ then $\mathrm{Sh}(z,1) = 1$.

(4) $U \triangleright V$ then

$$\mathrm{Sh}(1,1) : \mathrm{Sh}(G,Y)_V \longrightarrow \mathrm{Sh}(G,Y)_U$$

is Galois with group $U/V(Z(G)(\mathbb{Q}) \cap U)$, where $u \in U$ acts as $\mathrm{Sh}(u,1)$.

(5) If $x \in \lim_{\leftarrow V} \mathrm{Sh}(G,Y)_V(\mathbb{C})$ then the image of $G(\mathbb{A}^\infty)x$ is dense in $\mathrm{Sh}(G,Y)_U(\mathbb{C})$, for any U.

This implies the following: If $T \subset G$ is a maximal torus defined over $\mathbb{Q}$ with $T^{\mathrm{ad}}(\mathbb{R})$ is compact, if $i : T \hookrightarrow G$ denotes this embedding, and if $\mu \in Y$ factors through T (such a μ always exists); then

$$\bigcup_{g \in G(\mathbb{A}^\infty)} \mathrm{Sh}(g,i)(\mathrm{Sh}(T,\{\mu\})_{g^{-1}Ug \cap T(\mathbb{A}^\infty)}(\mathbb{C}))$$

is dense in $\mathrm{Sh}(G,\mathbb{C})_U(\mathbb{C})$.

(6) The group of automorphisms of the variety $\mathrm{Sh}(G,Y)_U$ is finite.

(For most of this see sections 1.8 and 1.14 of [D1]. For the uniqueness of the quasi-projective algebraic structure on $\mathrm{Sh}(G,Y)_U$ see [B2]. Item (5) above follows from the density of $G(\mathbb{Q})\mu$ in Y for any $\mu \in Y$, or even from the density of $G(\mathbb{Q})$ in $G(\mathbb{R})$. Item (6) follows from lemma 2.6.3 of [Ma]. (See also lemma 2.2 of [Mi2].))

As best we understand the main theorem of [Mi1] (proving a conjecture of Langlands from [L]), it asserts the following:

Theorem 3.2 (Milne)**.** *Suppose that (G,Y) is an NCF Deligne Shimura datum, that $E/\mathbb{Q}$ is a finite Galois extension, and that $\mu \in Y$ is an E-special point. Suppose also that $\tau \in \mathrm{Aut}(\mathbb{C})$ and choose $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ above $\tau|_{\mathbb{C}^{\mathrm{alg}}}$. Then there are unique morphisms*

$$\Phi(\tau,\mu,\alpha) : \tau\mathrm{Sh}(G,Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)}$$

such that

$$\Phi(\tau,\mu,\alpha)({}^\tau[(1,\mu)]) = (1, {}^\tau\mu)$$

and

$$\Phi(\tau,\mu,\alpha) \circ {}^\tau\mathrm{Sh}(g,1) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),1) \circ \Phi(\tau,\mu,\alpha)$$

for all $g \in G(\mathbb{A}^\infty)$.

If μ_1 and $\mu_2 \in Y$ are two E-special points, then

$$\Phi(\tau,\mu_2,\alpha) = \mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2}, \mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}}) \circ \Phi(\tau,\mu_1,\alpha).$$

(Note that the right hand side is unchanged if $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}$ is replaced by $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}\beta$ with $\beta \in {}^{\tau,\mu_2,\alpha}G(\mathbb{Q})$, and so the ambiguity in $\gamma_{E,\tau,\alpha,\mu_1,\mu_2}$ is unimportant.)

From these assertions the following additional formulae are easily deduced:

(1) If $\gamma \in S_{E,\mathbb{C}}(E)$ then $\Phi(\tau,\mu,\alpha\gamma) = \mathrm{Sh}(1, \mathrm{conj}_{(\mathrm{ad}\,\widetilde{\mu})(\gamma)^{-1}})\Phi(\tau,\mu,\alpha)$.

(2) If $f : (G_1, Y_1) \to (G_2, Y_2)$ and $g \in G_2(\mathbb{A}^\infty)$ and $\mu_1 \in Y_1$ is an E-special point, then $\Phi(\tau, f \circ \mu_1, \alpha) \circ {}^\tau \mathrm{Sh}(g, f) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f\circ\mu}}(g), {}^{\tau,\mu_1,\alpha} f) \circ \Phi(\tau, \mu_1, \alpha)$.
(3) $\Phi(\tau_1\tau_2, \mu, \alpha_1\alpha_2) = \Phi(\tau_1, {}^{\tau_2}\mu, \alpha_1) \circ {}^{\tau_1}\Phi(\tau_2, \mu, \alpha_2)$.
(4) If $G = T$ is a torus then $\Phi(\tau, \mu, \alpha) \circ \tau \circ \Pi_{T,\{\mu\}} = \Pi_{T,\{{}^\tau\mu\}}$.

We note that the simple composition relation (3) really depends on making $\Phi(\tau, \mu, \alpha)$ depend on the choice of α and not just of τ. It would seem that to make it depend on τ alone one would need to find a section to

$$\widetilde{S}_{E,\mathbb{C}}(E) \twoheadrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}} \cap \mathbb{C}/\mathbb{Q}),$$

i.e. a rational section not an adelic one. This is the reason we choose not to follow Langlands, but to make the choice of α explicit.

3.4. **Removing the NCF-condition.** We start with the following lemma.

Lemma 3.3. *Suppose that (G, Y) is a Shimura datum. Suppose also that $H \subset G$ is a normal connected reductive subgroup such that $(G/H)(\mathbb{R})$ is compact and the image of one, and hence every, $\mu \in Y$ in $(G/H)(\mathbb{R})$ is trivial. We will write i for the inclusion $H \hookrightarrow G$. Also suppose that U is a sufficiently small open compact subgroup of $G(\mathbb{A}^\infty)$.*

(1) Then Y is a single $H(\mathbb{R})$-conjugacy class so that (H, Y) is also a Shimura datum.
(2) $G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ has finite cardinality.
(3) $(G/H)(\mathbb{Q}) \cap \mathrm{Im}\,(U \to (G/H)(\mathbb{A}^\infty)) = \{1\}$.
(4) $G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) = \coprod_{h \in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} H(\mathbb{Q})\backslash(H(\mathbb{A}^\infty)/(hUh^{-1} \cap H(\mathbb{A}^\infty)) \times Y)h$.
(5) $\mathrm{Sh}(G, Y)_U = \coprod_{h \in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} \mathrm{Sh}(H, Y)_{hUh^{-1} \cap H(\mathbb{A}^\infty)}$, where we map

$$\mathrm{Sh}(H, Y)_{hUh^{-1} \cap H(\mathbb{A}^\infty)} \hookrightarrow \mathrm{Sh}(G, Y)_U$$

via $\mathrm{Sh}(h^{-1}, i)$.

Proof: The exact sequence

$$(0) \longrightarrow H^{\mathrm{ad}} \longrightarrow G^{\mathrm{ad}} \longrightarrow (G/H)^{\mathrm{ad}} \longrightarrow (0)$$

has a unique splitting in which $(G/H)^{\mathrm{ad}}$ lifts to a normal subgroup of G^{ad}. Write H' for the pre-image in G of $(G/H)^{\mathrm{ad}} \subset H^{\mathrm{ad}} \times (G/H)^{\mathrm{ad}} = G^{\mathrm{ad}}$, so that $(H')^{\mathrm{ad}} \xrightarrow{\sim} G/H$. Note that $H'(\mathbb{R}) \twoheadrightarrow (G/H)(\mathbb{R})$ (as $(G/H)(\mathbb{R})$ is compact) and acts trivially on Y. If $\mu, \mu' \in Y$ then $\mu' = \mathrm{conj}_g \circ \mu$ for some $g \in G(\mathbb{R})$. Let $h \in H'(\mathbb{R})$ have the same image as g in $(G/H)(\mathbb{R})$. Thus $gh^{-1} \in H(\mathbb{R})$ and $\mathrm{conj}_{gh^{-1}} \circ \mu = \mu'$. The first part of the lemma follows.

The set

$$G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U = G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A})/UG(\mathbb{R})$$

is finite by theorem 5.1 of [PR].

For the third part we see that $(G/H)(\mathbb{Q}) \cap \mathrm{Im}\,(U \to (G/H)(\mathbb{A}^\infty))$ is finite (because $(G/H)(\mathbb{R})$ is compact) and hence $\{1\}$ because U is sufficiently small.

For the fourth part, first note that

$$G(\mathbb{Q})\backslash(G(\mathbb{A}^\infty)/U \times Y) = \coprod_{h\in G(\mathbb{Q})H(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} G(\mathbb{Q})\backslash(G(\mathbb{Q})H(\mathbb{A}^\infty)hU/U \times Y).$$

Next suppose that for $g_1, g_2 \in H(\mathbb{A}^\infty)$ and $\mu_1, \mu_2 \in Y$ we have

$$\gamma(g_1hu, \mu_1) = (g_2h, \mu_2),$$

for some $\gamma \in G(\mathbb{Q})$ and $u \in U$. Then we see that the image of γ in $(G/H)(\mathbb{Q})$ lies in $(G/H)(\mathbb{Q}) \cap \mathrm{Im}\,(hUh^{-1} \to (G/H)(\mathbb{A}^\infty)) = \{1\}$. Thus $\gamma \in H(\mathbb{Q})$ and $huh^{-1} \in H(\mathbb{A}^\infty)$. We conclude that

$$H(\mathbb{Q})\backslash(H(\mathbb{A}^\infty)/(hUh^{-1} \cap H(\mathbb{A}^\infty)) \times Y) \xrightarrow{h} G(\mathbb{Q})\backslash(G(\mathbb{Q})H(\mathbb{A}^\infty)hU/U \times Y)$$

is an isomorphism, and the third part of the lemma follows. The fifth part follows from the fourth and the uniqueness assertion in section 3.3. □

Suppose that (G, Y) is a Deligne Shimura datum. We have $G^{\mathrm{ad}} = G^{\mathrm{ad},nc} \times G^{\mathrm{ad},c}$, where $G^{\mathrm{ad},c}(\mathbb{R})$ is compact, but if H is any simple factor of $G^{\mathrm{ad},nc}/\mathbb{Q}$, then $H(\mathbb{R})$ is not compact. We will write G^{nc} (resp. G^c) for the connected component of the identity of $\ker(G \twoheadrightarrow G^{\mathrm{ad},c})$ (resp. $\ker(G \twoheadrightarrow G^{\mathrm{ad},nc})$) and $\overline{G}^{nc}$ (resp. $\overline{G}^c$) for G/G^c (resp. G/G^{nc}). Thus

$$G^c \twoheadrightarrow \overline{G}^c \twoheadrightarrow G^{\mathrm{ad},c} \xrightarrow{\sim} G^{c,\mathrm{ad}}$$

and

$$G^{nc} \twoheadrightarrow \overline{G}^{nc} \twoheadrightarrow G^{\mathrm{ad},nc} \xrightarrow{\sim} G^{nc,\mathrm{ad}},$$

where the central maps have finite central kernels. We also have $Z(G^c) = Z(G) \cap G^c$ and $Z(G^{nc}) = Z(G) \cap G^{nc}$. Moreover G^c and G^{nc} centralize each other. (Indeed if we let G^c act on G^{nc} by conjugation, we see that, given $h \in G^{nc}$, there is a character $\chi_h : G^c \to Z(G) \cap G^{nc}$ such that $\mathrm{conj}_g(h) = \chi_h(g)h$. The character χ_h must factor through $C(G^c)$, but is trivial on $Z(G) \cap G^c \twoheadrightarrow C(G^c)$. Thus $\chi_h = 1$ and G^c centralizes h as desired.) We have an exact sequence

$$(0) \longrightarrow Z(G^c) \cap Z(G^{nc}) \longrightarrow G^{nc} \times G^c \longrightarrow G \longrightarrow (0).$$

Note $(G/G^{nc})(\mathbb{R})$ is compact and hence connected. Thus $G^c(\mathbb{R}) \twoheadrightarrow (G/G^{nc})(\mathbb{R})$.

If $\mu \in Y$ then the composition of μ with $G \to G^{\mathrm{ad},c}$ takes -1 to 1 and hence factors through the squaring map $\mathbb{G}_m \to \mathbb{G}_m$. As this composition is miniscule we see that it must actually be trivial, i.e. $\mu \in X_*(G^{nc})$ and G^c centralizes μ. By lemma 3.3 (G^{nc}, Y) is a NCF-Shimura datum. Write i for the map $G^{nc} \hookrightarrow G$, so that $i : (G^{nc}, Y) \to (G, Y)$. Further by lemma 3.3 we have

$$\mathrm{Sh}(G, Y)_U = \coprod_{h\in G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U} \mathrm{Sh}(G^{nc}, Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)},$$

where $G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ is finite, and where

$$\mathrm{Sh}(h^{-1}, i) : \mathrm{Sh}(G^{nc}, Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} \hookrightarrow \mathrm{Sh}(G, Y)_U.$$

If $h' = g\gamma hu$ with $g \in G^{nc}(\mathbb{A}^\infty)$, $\gamma \in G(\mathbb{Q})$ and $u \in U$, then

$$\begin{array}{ccc} \mathrm{Sh}(G^{nc}, Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} & & \\ & \searrow \mathrm{Sh}(h^{-1}, i) & \\ \mathrm{Sh}(g, \mathrm{conj}_\gamma) \downarrow \wr & & \mathrm{Sh}(G, Y)_U \\ & \nearrow \mathrm{Sh}((h')^{-1}, i) & \\ \mathrm{Sh}(G^{nc}, Y)_{h'U(h')^{-1}\cap G^{nc}(\mathbb{A}^\infty)} & & \end{array}$$

commutes.

Now suppose that $f : (G_1, Y_1) \to (G_2, Y_2)$ is a morphism of Shimura data, that $U_i \subset G_i(\mathbb{A}^\infty)$ are sufficiently small open compact subgroups and that $g \in G_2(\mathbb{A}^\infty)$ such that $gf(U_1)g^{-1} \subset U_2$. Note that $f : G_1^{nc} \to G_2^{nc}$. If $h \in G(\mathbb{A}^\infty)$ then

$$\begin{array}{ccc} \mathrm{Sh}(G_1^{nc}, Y_1)_{hU_1h^{-1}\cap G_1^{nc}(\mathbb{A}^\infty)} & \stackrel{\mathrm{Sh}(h^{-1}, i_1)}{\hookrightarrow} & \mathrm{Sh}(G_1, Y_1)_{U_1} \\ \mathrm{Sh}(1, f) \downarrow & & \downarrow \mathrm{Sh}(g, f) \\ \mathrm{Sh}(G_2^{nc}, Y_2)_{f(h)g^{-1}U_2(f(h)g^{-1})^{-1}\cap G_2^{nc}(\mathbb{A}^\infty)} & \stackrel{\mathrm{Sh}(gf(h^{-1}), i_2)}{\hookrightarrow} & \mathrm{Sh}(G_2, Y_2)_{U_2} \end{array}$$

commutes.

Our next aim is to extend theorem 3.2 to this setting. So suppose that (G, Y) is a Shimura datum and that $\mu \in Y$ is an E-special point. Suppose also that $\tau \in \mathrm{Aut}\,(\mathbb{C})$ and $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ lies above $\tau|_{\mathbb{C}^{\mathrm{alg}}}$.

Note that $\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ as defined for G equal those defined for G^{nc}. Thus we will denote them with the same symbol. Hence $\overline{G}^c = \overline{{}^{\tau,\mu,\alpha}G}^c$ and ${}^{\tau,\mu,\alpha}(G^{nc}) = ({}^{\tau,\mu,\alpha}G)^{nc}$. We claim that the images of $G(\mathbb{Q})$ and ${}^{\tau,\mu,\alpha}G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$ are equal, from which it follows that

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)) = {}^{\tau,\mu,\alpha}G(\mathbb{Q})^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty),$$

and hence that $\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}$ gives a bijection

$$G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U \stackrel{\sim}{\longrightarrow} {}^{\tau,\mu,\alpha}G(\mathbb{Q})^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)\backslash{}^{\tau,\mu,\alpha}G(\mathbb{A}^\infty)/\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U).$$

To prove the claim suppose that $\gamma \in G(\mathbb{Q})$. Then we have $\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu} \in G^{nc}(E)$ satisfying

$$\begin{array}{cl} & \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mathrm{conj}_{\gamma^{-1}}\circ\mu}(\sigma)\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^{-1} \\ = & \gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^\sigma\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\circ\mu}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}(\sigma)^{-1} \end{array}$$

i.e.

$$\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} = \mathrm{conj}_{\phi^{\mathrm{ad}}_{E,\tau,\alpha,\mu}(\sigma)}({}^\sigma(\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu})).$$

Hence

$$\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \in {}^{\tau,\mu,\alpha}G(\mathbb{Q})$$

and has the same image in $\overline{G}^c(E)$ as γ. Thus the image of $G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$ is contained in the image of ${}^{\tau,\mu,\alpha}G(\mathbb{Q})$ in $\overline{G}^c(\mathbb{Q})$. Using the identification ${}^{\tau^{-1},{}^\tau\mu,\alpha^{-1}}({}^{\tau,\mu,\alpha}G) = G$ we get the reverse inclusion.

Now define

$$\Phi(\tau,\mu,\alpha) : \tau\mathrm{Sh}(G,Y)_U \stackrel{\sim}{\longrightarrow} \mathrm{Sh}({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)}$$

to be the disjoint union over $h \in G(\mathbb{Q})G^{nc}(\mathbb{A}^\infty)\backslash G(\mathbb{A}^\infty)/U$ of the maps $\Phi(\tau,\mu,\alpha)$:

$$\tau\mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)} \stackrel{\sim}{\to} \mathrm{Sh}({}^{\tau,\mu,\alpha}G^{nc}, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)^{-1}\cap {}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)}.$$

From the claim above we see that $\Phi(\tau,\mu,\alpha)$ is an isomorphism. We must check it is independent of the choice of coset representatives h. For this suppose that $h' = g\gamma h u$ with $g \in G^{nc}(\mathbb{A}^\infty)$ and $\gamma \in G(\mathbb{Q})$ and $u \in U$. Then

$$\begin{array}{rcl}\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h') &=& (\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1})(\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}) \\ && \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(u),\end{array}$$

with $\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \in {}^{\tau,\mu,\alpha}G(\mathbb{Q})$ and

$$\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1} \in {}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty).$$

Thus what we must show is that

$$\Phi(\tau,\mu,\alpha)\circ {}^\tau\mathrm{Sh}(g,\mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}, \mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}})\circ\Phi(\tau,\mu,\alpha).$$

However

$$\begin{array}{rl} & \Phi(\tau,\mu,\alpha)\circ {}^\tau\mathrm{Sh}(g,\mathrm{conj}_\gamma) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),\mathrm{conj}_\gamma)\circ\Phi(\tau,\mathrm{conj}_{\gamma^{-1}}\mu,\alpha) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),\mathrm{conj}_\gamma)\circ\mathrm{Sh}(b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}, \mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}})\circ\Phi(\tau,\mu,\alpha).\end{array}$$

Thus we are reduced to checking that

$$\begin{array}{rl} & \mathrm{Sh}(\gamma b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}, \mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}}) \\ = & \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(\gamma)\gamma^{-1}_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}\gamma^{-1}, \mathrm{conj}_{\gamma\gamma_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu}}).\end{array}$$

This is clear because

$$\begin{array}{rl} & \gamma b_{E,\tau,\alpha,\mu,\mathrm{conj}_{\gamma^{-1}}\mu} \\ = & \gamma\mathrm{conj}_{\gamma^{-1}}(\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu})\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \\ = & \widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}\gamma\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu} \\ = & \mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(\gamma).\end{array}$$

We certainly have

$$\Phi(\tau,\mu,\alpha)(\mu,1) = ({}^\tau\mu,1).$$

If $gU_1g^{-1}\subset U_2$, we claim that

$$\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(g,1) = \mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),1)\circ\Phi(\tau,\mu,\alpha),$$

as maps

$${}^\tau\mathrm{Sh}(G,Y)_{U_1} \to \mathrm{Sh}({}^{\tau,\mu,\alpha}G, {}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U_2)}.$$

However both sides when restricted to $\mathrm{Sh}(G,Y)_{hU_1h^{-1}\cap G^{nc}(\mathbb{A}^\infty)}$ are just $\Phi(\tau,\mu,\alpha)$ taking

$$ {}^\tau\mathrm{Sh}(G^{nc},Y)_{hU_1h^{-1}\cap G^{nc}(\mathbb{A}^\infty)} \longrightarrow \mathrm{Sh}({}^{\tau,\mu,\alpha}G^{nc},{}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}hg^{-1}(U_2)\cap {}^{\tau,\mu,\alpha}G^{nc}(\mathbb{A}^\infty)}. $$

Now suppose that μ_1 and μ_2 are special in Y and defined over E. We claim that

$$ \Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha) $$

as maps

$$ {}^\tau\mathrm{Sh}(G,Y)_U \longrightarrow \mathrm{Sh}({}^{\tau,\mu_2,\alpha}G,{}^{\tau,\mu_2,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}U}. $$

To verify this, we must show that if $h\in G(\mathbb{A}^\infty)$, then

$$ \Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,b_1,b_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2},1)\circ\mathrm{Sh}(1,\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha) $$

as maps from ${}^\tau\mathrm{Sh}(G^{nc},Y)_{hUh^{-1}\cap G^{nc}(\mathbb{A}^\infty)}$ to

$$ \mathrm{Sh}({}^{\tau,\mu_2,\alpha}G^{nc},{}^{\tau,\mu_2,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(h)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(U)\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu_2}}(h)^{-1}\cap{}^{\tau,\mu_2,\alpha}G^{nc}(\mathbb{A}^\infty)}. $$

However this equality is part of theorem 3.2.

Thus Milne's theorem 3.2 remains true without the NCF hypothesis. As noted immediately after the statement of that theorem, this allows us to conclude:

Theorem 3.4. *Suppose that $E/\mathbb{Q}$ is a finite Galois extension, that (G,Y) is a Deligne Shimura datum, and that $\mu\in Y$ is an E-special point. Suppose also that $\tau\in\mathrm{Aut}\,(\mathbb{C})$ and choose $\alpha\in\widetilde{S}_{E,\mathbb{C},\tau}$. Then there is a unique morphism*

$$ \Phi(\tau,\mu,\alpha):\tau\mathrm{Sh}(G,Y)_U \xrightarrow{\sim} \mathrm{Sh}({}^{\tau,\mu,\alpha}G,{}^{\tau,\mu,\alpha}Y)_{\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(U)} $$

such that

$$ \Phi(\tau,\mu,\alpha)(\mu,1)=({}^\tau\mu,1) $$

and

$$ \Phi(\tau,\mu,\alpha)\circ\mathrm{Sh}(g,1)=\mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,\mu}}(g),1)\circ\Phi(\tau,\mu,\alpha) $$

for all $g\in G(\mathbb{A}^\infty)$. Moreover:

1. *If $\gamma\in S_{E,\mathbb{C}}(E)$ then $\Phi(\tau,\mu,\alpha\gamma)=\mathrm{Sh}(1,\mathrm{conj}_{\widetilde{\mu}(\gamma)^{-1}})\Phi(\tau,\mu,\alpha)$.*
2. *If $f:(G_1,Y_1)\to(G_2,Y_2)$ and $g\in G_2(\mathbb{A}^\infty)$ and $\mu_1\in Y_1$ is a special point defined over the image of E in $\mathbb{C}$, then*
$$ \Phi(\tau,f\circ\mu_1,\alpha)\circ{}^\tau\mathrm{Sh}(g,f)=\mathrm{Sh}(\mathrm{conj}_{b^{\mathrm{ad}}_{E,\tau,\alpha,f\circ\mu}}(g),{}^{\tau,\mu_1,\alpha}f)\circ\Phi(\tau,\mu_1,\alpha). $$
3. *$\Phi(\tau_1\tau_2,\mu,\alpha_1\alpha_2)=\Phi(\tau_1,{}^{\tau_2}\mu,\alpha_1)\circ{}^{\tau_1}\Phi(\tau_2,\mu,\alpha_2)$.*
4. *If $G=T$ is a torus then $\Phi(\tau,\mu,\alpha)\circ\tau\circ\Pi_{T,\{\mu\}}=\Pi_{T,\{{}^\tau\mu\}}$.*
5. *If μ_1 and μ_2 are two such special points defined over E, then*
$$ \Phi(\tau,\mu_2,\alpha)=\mathrm{Sh}(b_{E,\tau,\alpha,\mu_1,\mu_2}\gamma^{-1}_{E,\tau,\alpha,\mu_1,\mu_2},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu_1,\mu_2}})\circ\Phi(\tau,\mu_1,\alpha). $$

3.5. **Reformulation of Milne's theorem.** We now state and prove our first main theorem, which is a reformulation of Milne's theorem.

Theorem 3.5. *Suppose that $E/\mathbb{Q}$ is a finite Galois extension and that $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$. If (G,Y) is a Shimura datum with E acceptable for G, if $(\tau,\phi,b) \in \mathrm{Conj}_{E,\mathfrak{a}}(G,Y)$ and if U is a sufficiently small open compact subgroup of $G(\mathbb{A}^\infty)$, then there is an isomorphism*

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) : {}^\tau\mathrm{Sh}(G,Y)_U \xrightarrow{\sim} \mathrm{Sh}^{(\tau,\phi,b)}(G,Y)_{bUb^{-1}}$$

with the following properties.

1. $\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) \circ {}^\tau\mathrm{Sh}(g,1) = \mathrm{Sh}(bgb^{-1},1) \circ \Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)$.
2. $\mathrm{Sh}(1,f) \circ \Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) = \Phi_{E,\mathfrak{a}^+}(\tau,f\circ\phi,f(b)) \circ {}^\tau\mathrm{Sh}(1,f)$.
3. *If $\delta \in G(E)$ and $h \in G(\mathbb{A}^\infty)$, then* $\Phi_{E,\mathfrak{a}^+}(\tau,{}^\delta\phi,\delta bh) = \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ {}^\tau\mathrm{Sh}(h,1)$.
4. *If $(\tau_1,\phi_1,b_1) \in \mathrm{Conj}_{E,\mathfrak{a}^+}{}^{(\tau_2,c_2,b_2)}(G,Y)$ and $(\tau_2,\phi_2,b_2) \in \mathrm{Conj}_{E,\mathfrak{a}^+}(G,Y)$, then*
$$\Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2,\phi_1\phi_2,b_1b_2) = \Phi_{E,\mathfrak{a}^+}(\tau_1,\phi_1,b_1) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2,\phi_2,b_2).$$
5. *Suppose that $G = T$ is a torus, that $\mu \in X_*(T)(\mathbb{C})$ and that $(\tau,\phi,b) \in \mathrm{Conj}_{E,\mathfrak{a}}(T,\{\mu\})$. Then*
$$b^{-1}\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \in T(\mathbb{A}^\infty)/\overline{T(\mathbb{Q})} \subset T(\mathbb{A}_E^\infty)/\overline{T(\mathbb{Q})}T(E).$$
Moreover
$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) \circ \tau \circ \Pi_{T,\{\mu\}} = \mathrm{Sh}(b\overline{b}^{-1}_{\mathfrak{a}^+,\infty,\mu,\tau},1) \circ \Pi_{T,\{{}^\tau\mu\}}.$$
In the special case that τ fixes the image of E in $\mathbb{C}$, then $\Pi^{-1}_{T,\{{}^\tau\mu\}}\circ\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)\circ \tau\circ\Pi_{T,\{\mu\}}$ equals multiplication by
$$b^{-1}\prod_{\rho:E\hookrightarrow\mathbb{C}} ({}^{\rho^{-1}}\mu)(\mathrm{Art}_E^{-1}\tau^{\widetilde{\rho}})^{-1},$$
where $\widetilde{\rho}$ is any extension of ρ to E^{ab}.
6. *Suppose that $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with ${}^t\inf_{D/E}\mathfrak{a}^+ = \eta_{D/E,*}\mathfrak{a}_D^+$. Then*
$$\Phi_{D,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau,\phi,b)) = \Phi_{E,\mathfrak{a}^+}(\tau,\phi,b).$$
7. *If $\mu \in Y$ is an E-special point and if $\alpha \in \widetilde{S}_{E,\mathbb{C}}(E)$ lifts $\tau|_{\mathbb{C}^{\mathrm{alg}}}$, then*
$$\Phi_{E,\mathfrak{a}^+}(\tau,\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu},\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}) = \Phi(\tau,\mu,\alpha).$$

Proof: Suppose that $T \subset G$ is a maximal torus defined over $\mathbb{Q}$ such that $T^{\mathrm{ad}}(\mathbb{R})$ is compact and T is split by E. Then we may choose $\mu \in Y$ that factors through T. It will be E-special. Choosing α as in part (7) of the theorem, we may find $\delta \in G(E)$ and $h \in G(\mathbb{A}^\infty)$ such that

$$(\tau,\phi,b) = (\tau,{}^\delta\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu},\delta\widetilde{b}_{K,\mathfrak{a}^+,\tau,\alpha,\mu}h).$$

Then we are forced to set

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b)=\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1).$$

We must check that this is a good definition.

If we replace $\widetilde{b}_{K,\mathfrak{a}^+,\tau,\alpha,\mu}$ by $\widetilde{\mu}_{Y,E}(\gamma z)\widetilde{b}_{K,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ by ${}^{\widetilde{\mu}_{Y,E}(\gamma)}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$, with $\gamma\in R^1_{E,\mathbb{C}}(E)$ and $z\in\overline{R^1_{E,\mathbb{C}}(\mathbb{Q})}$; then δ is replaced by $\delta\widetilde{\mu}_{Y,E}(\gamma)^{-1}$ and h is replaced by $h\widetilde{\mu}_{Y,E}(z)^{-1}$. But $\mathrm{Sh}(1,\mathrm{conj}_{\delta\widetilde{\mu}_{Y,E}(\gamma)^{-1}})=\mathrm{Sh}(1,\mathrm{conj}_\delta)$ and $\mathrm{Sh}(h\widetilde{\mu}_{Y,E}(z)^{-1},1)=\mathrm{Sh}(h,1)$, and so the definition is independent of the choice of $\widetilde{b}_{K,\mathfrak{a}^+,\tau,\alpha,\mu}$.

If $\widetilde{\gamma}\in{}^{\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}}G(\mathbb{Q})$ then

$$\begin{array}{rl} & \mathrm{Sh}(1,\mathrm{conj}_{\delta\gamma})\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(b^{-1}\gamma^{-1}bh,1)\\ = & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\mathrm{Sh}(\gamma,1)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(b^{-1}\gamma^{-1}bh,1)\\ = & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1),\end{array}$$

and so the definition is independent of the choice of δ and h.

If we replace α by $\alpha\gamma^{-1}$ with $\gamma\in S_{E,\mathbb{C}}(E)$, then there is a lift $\widetilde{\gamma}\in R_{E,\mathbb{C}}(E)$ of γ such that $\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}={}^{\widetilde{\mu}(\widetilde{\gamma})}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}=\widetilde{\mu}(\widetilde{\gamma})\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and so

$$(\tau,\phi,b)=(\tau,{}^{\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu},\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha\gamma^{-1},\mu}h).$$

Then, because $\Phi(\tau,\mu,\alpha\gamma^{-1})=\mathrm{Sh}(1,\mathrm{conj}_{\widetilde{\gamma}})\circ\Phi(\tau,\mu,\alpha)$, we see that

$$\mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1)=\mathrm{Sh}(1,\mathrm{conj}_{\delta\widetilde{\mu}(\widetilde{\gamma})^{-1}})\circ\Phi(\tau,\mu,\alpha\gamma^{-1})\circ{}^\tau\mathrm{Sh}(h,1),$$

and our definition is independent of the choice of α.

Finally if we replace μ by μ', then

$$(\tau,\phi,b)=(\tau,{}^{\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'},(\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'})\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}(\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}h)).$$

We must check that

$$\begin{array}{rl} & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(h,1)\\ = & \mathrm{Sh}(1,\mathrm{conj}_{\delta\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}})\circ\Phi(\tau,\mu',\alpha)\circ{}^\tau\mathrm{Sh}(\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}\gamma_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}h,1),\end{array}$$

or that

$$\begin{array}{rl} & \Phi(\tau,\mu,\alpha)\circ{}^\tau\mathrm{Sh}(\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'},1)\\ = & \mathrm{Sh}(1,\mathrm{conj}_{\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}})\circ\mathrm{Sh}(b_{E,\tau,\alpha,\mu,\mu'}\gamma^{-1}_{E,\tau,\alpha,\mu,\mu'},\mathrm{conj}_{\gamma_{E,\tau,\alpha,\mu,\mu'}})\circ\Phi(\tau,\mu,\alpha),\end{array}$$

or even that

$$\begin{array}{rl} & \mathrm{Sh}(\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu'}\widetilde{b}^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu},1)\circ\Phi(\tau,\mu,\alpha)\\ = & \mathrm{Sh}(\gamma^{-1}_{E,\mathfrak{a}^+,\tau,\alpha,\mu,\mu'}b_{E,\tau,\alpha,\mu,\mu'},1)\circ\Phi(\tau,\mu,\alpha),\end{array}$$

which is true.

Having checked that our definition is good we must check the desired properties. Property (7) is part of the definition, while property (3) follows easily from the definition.

Properties (1) and (2) are true for

$$(\tau, \phi, b) = (\tau, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}),$$

because we can take $\widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,f\circ\mu} = f \circ \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}$ and $\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,f\circ\mu} = f(\widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu})$. To check that they remain true for all (τ, ϕ, b), it suffices to check that if they are true (τ, ϕ, b) then they are also true for $(\tau, {}^{\delta}\phi, \delta bh)$. However we have

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau, {}^{\delta}\phi, \delta bh) \circ {}^{\tau}\mathrm{Sh}(g, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_{\delta}) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ {}^{\tau}\mathrm{Sh}(hg, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_{\delta}) \circ \mathrm{Sh}(bhgh^{-1}b^{-1}, 1) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ {}^{\tau}\mathrm{Sh}(h, 1) \\ = & \mathrm{Sh}(\mathrm{conj}_{\delta bh}(g), 1) \circ \Phi_{E,\mathfrak{a}^+}(\tau, {}^{\delta}\phi, \delta bh) \end{array}$$

and

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau, f \circ {}^{\delta}\phi, f(\delta bh)) \circ {}^{\tau}\mathrm{Sh}(1, f) \\ = & \mathrm{Sh}(1, \mathrm{conj}_{f(\delta)}) \circ \Phi_{E,\mathfrak{a}^+}(\tau, f \circ \phi, f(b)b) \circ {}^{\tau}\mathrm{Sh}(f(h), 1) \circ {}^{\tau}\mathrm{Sh}(1, f) \\ = & \mathrm{Sh}(1, \mathrm{conj}_{f(\delta)}) \circ \Phi_{E,\mathfrak{a}^+}(\tau, f \circ \phi, f(b)b) \circ {}^{\tau}\mathrm{Sh}(1, f) \circ {}^{\tau}\mathrm{Sh}(h, 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_{f(\delta)}) \circ \mathrm{Sh}(1, f) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ {}^{\tau}\mathrm{Sh}(h, 1) \\ = & \mathrm{Sh}(1, f) \circ \mathrm{Sh}(1, \mathrm{conj}_{\delta}) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ {}^{\tau}\mathrm{Sh}(h, 1) \\ = & \mathrm{Sh}(1, f) \circ \Phi_{E,\mathfrak{a}^+}(\tau, {}^{\delta}\phi, \delta bh). \end{array}$$

Similarly property (5) is true in the case

$$(\tau, \phi, b) = (\tau, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau,\alpha,\mu}).$$

On the other hand if the claim is true for (τ, ϕ, b), then

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau, {}^{\delta}\phi, \delta bh) \circ \tau \circ \Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(1, \mathrm{conj}_{\delta}) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ {}^{\tau}\mathrm{Sh}(h, 1) \circ \tau \circ \Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(h, 1) \circ \Phi_{E,\mathfrak{a}^+}(\tau, \phi, b) \circ \tau \circ \Pi_{T,\{\mu\}} \\ = & \mathrm{Sh}(h, 1) \circ \mathrm{Sh}(b\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}^{-1}, 1) \circ \Pi_{T,\{{}^{\tau}\mu\}} \\ = & \mathrm{Sh}(\delta bh\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}^{-1}, 1) \circ \Pi_{T,\{{}^{\tau}\mu\}}, \end{array}$$

and so it is also true for $(\tau, {}^{\delta}\phi, \delta bh)$.

That property (6) is true in the case $G = T$ is a torus follows from property (5) because $\inf_{D/E,t}(\tau, \phi, b) = (\tau, \inf_{D/E,t}\phi, \boldsymbol{\nu}_{\phi}(t)b)$ and

$$\overline{b}_{\mathfrak{a}_D^+,\infty,\mu,\tau} = \overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \prod_{\rho:E\hookrightarrow\mathbb{C}} (({}^{\rho^{-1}}\mu) \circ (\pi_{w(\rho)}/\pi_{w(\tau\rho)}))(t) = \overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau}\boldsymbol{\nu}_{\phi}(t).$$

Now consider the general case. Because $\inf_{D/E,t}(\gamma, h)(\tau, \phi, b) = (\gamma, h)\inf_{D/E,t}(\tau, \phi, b)$, the assertion will be true for (τ, ϕ, b) if and only if it is true for $(\gamma, h)(\tau, \phi, b)$. Choose a maximal torus $T \subset G$ defined over $\mathbb{Q}$ and split by E such that $T^{\mathrm{ad}}(\mathbb{R})$ is compact. Also choose $\mu \in Y$ which factors through T and let i denote the canonical embedding $i : T \hookrightarrow G$. Also choose $(\tau, \phi, b) \in \mathrm{Conj}_{E,\mathfrak{a}}(T, \{\mu\})$. It will suffice to prove that

$$\Phi_{E,\mathfrak{a}^+}(\tau, i \circ \phi, i(b)) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau, i \circ \phi, i(b))).$$

Because

$$\bigcup_{g\in G(\mathbb{A}^\infty)} \mathrm{Sh}(g,i)(\mathrm{Sh}(T,\{\mu\})_{g^{-1}Ug\cap T(\mathbb{A}^\infty)})$$

is Zariski dense in $\mathrm{Sh}(G,Y)_U$, it even suffices to check that

$$\Phi_{E,\mathfrak{a}^+}(\tau, i\circ\phi, i(b))\circ {}^\tau\mathrm{Sh}(g,i) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau, i\circ\phi, i(b)))\circ {}^\tau\mathrm{Sh}(g,i)$$

for all $g\in G(\mathbb{A}^\infty)$. As $\mathrm{conj}_b(g)=\mathrm{conj}_{\boldsymbol{\nu}_\phi(t)b}(g)$ (because ϕ is basic) and $\inf_{D/E,t}(i\circ\phi) = i\circ\inf_{D/E,t}\phi$ and $i(\boldsymbol{\nu}_\phi(t))=\boldsymbol{\nu}_{i\circ\phi}(t)$; applying properties (1) and (2) we reduce to the equality

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) = \Phi_{E,\mathfrak{a}_D^+}(\inf_{D/E,t}(\tau,\phi,b)),$$

which we have already verified.

Finally we must check property (4). If

$$(\tau_1,\phi_1,b_1) = (\tau, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu})$$

and

$$(\tau_2,\phi_2,b_2) = (\tau, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu})$$

Then the result is true because for some $\beta\in Z(G)(E)$ we have

$$\widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} = {}^\beta\widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}$$

and

$$\widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\widetilde{b}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu} \equiv \beta\widetilde{b}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu} \bmod \overline{Z(G)(\mathbb{Q})},$$

so that

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\widetilde{\phi}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,{}^{\tau_2}\mu}\widetilde{b}_{E,\mathfrak{a}^+,\tau_2,\alpha_2,\mu}) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau_1\tau_2,\alpha_1\alpha_2,\mu}). \end{array}$$

Suppose now that property (4) holds for (τ_1,ϕ_1,b_1) and (τ_2,ϕ_2,b_2). Then it also holds for $(\tau_1, {}^\delta\phi_1, \delta b_1 h)$ and (τ_2,ϕ_2,b_2), because

$$\begin{array}{rl} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, ({}^\delta\phi_1)\phi_2, \delta b_1 h b_2) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, {}^\delta(\phi_1\phi_2), \delta b_1 b_2(b_2^{-1}hb_2)) \\ = & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2,\phi_1\phi_2,b_1b_2)\circ {}^{\tau_1\tau_2}\mathrm{Sh}(b_2^{-1}hb_2,1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi_{E,\mathfrak{a}^+}(\tau_1,\phi_1,b_1)\circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2,\phi_2,b_2)\circ {}^{\tau_1\tau_2}\mathrm{Sh}(b_2^{-1}hb_2,1) \\ = & \mathrm{Sh}(1,\mathrm{conj}_\delta)\circ\Phi_{E,\mathfrak{a}^+}(\tau_1,\phi_1,b_1)\circ {}^{\tau_1}\mathrm{Sh}(h,1)\circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2,\phi_2,b_2) \\ = & \Phi_{E,\mathfrak{a}^+}(\tau_1, {}^\delta\phi_1, \delta b_1 h)\circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2,\phi_2,b_2). \end{array}$$

Similarly, if the property holds for $(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \phi_1, \mathrm{conj}_{\delta^{-1}}(b_1))$ and (τ_2, ϕ_2, b_2), then it also holds for (τ_1, ϕ_1, b_1) and $(\tau_2, {}^{\delta}\phi_2, \delta b_2 h)$, because

$$\begin{aligned} & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, \phi_1({}^{\delta}\phi_2), b_1\delta b_2 h) \\ = \; & \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, {}^{\delta}((\mathrm{conj}_{\delta^{-1}} \circ \phi_1)\phi_2), \delta \mathrm{conj}_{\delta^{-1}}(b_1)b_2 h) \\ = \; & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1\tau_2, (\mathrm{conj}_{\delta^{-1}} \circ \phi_1)\phi_2, \mathrm{conj}_{\delta^{-1}}(b_1)b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = \; & \mathrm{Sh}(1, \mathrm{conj}_\delta) \circ \Phi_{E,\mathfrak{a}^+}(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \phi_1, \mathrm{conj}_{\delta^{-1}}(b_1)) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = \; & \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\mathrm{Sh}(1, \mathrm{conj}_\delta) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, \phi_2, b_2) \circ {}^{\tau_1\tau_2}\mathrm{Sh}(h, 1) \\ = \; & \Phi_{E,\mathfrak{a}^+}(\tau_1, \phi_1, b_1) \circ {}^{\tau_1}\Phi_{E,\mathfrak{a}^+}(\tau_2, {}^{\delta}\phi_2, \delta b_2 h). \end{aligned}$$

Note that

$$(\tau_1, \mathrm{conj}_{\delta^{-1}} \circ \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mu}, \mathrm{conj}_{\delta^{-1}}(\widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mu})) = (\tau_1, \widetilde{\phi}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mathrm{conj}_{\delta^{-1}}\circ\mu}, \widetilde{b}_{E,\mathfrak{a}^+,\tau_1,\alpha_1,\mathrm{conj}_{\delta^{-1}}\circ\mu}).$$

Thus property (4) follows in full generality. □

We remark that properties (1), (2) and (5) completely characterize the $\Phi_{E,\mathfrak{a}^+}(\tau, \phi, b)$.

4. Rational Shimura varieties

In this section we define, for any field L of characteristic 0, an explicit category $\mathrm{RSD}(L)$ (of 'rational Shimura data over L'), together with fully faithful functors

$$\mathrm{RSD}(\tau) : \mathrm{RSD}(L) \longrightarrow \mathrm{RSD}(L')$$

whenever $\tau : L \to L'$ is a map of fields; and construct functors

$$\mathrm{Sh}_L : \mathrm{RSD}(L) \longrightarrow \mathrm{QProj}_L$$

to the category of quasi-projective varieties over L together with natural isomorphisms

$$\Phi(\tau) : \tau \circ \mathrm{Sh}_L \stackrel{\sim}{\longrightarrow} \mathrm{Sh}_{L'} \circ \mathrm{RSD}(\tau)$$

whenever $\tau : L \to L'$ is a map of fields. Moroever we will have

$$\Phi(\tau' \circ \tau) = \Phi(\tau') \circ {}^{\tau'}\Phi(\tau).$$

4.1. **Rational Shimura data.** In this section we will define the categories $\mathrm{RSD}(L)$, for L a field of characteristic 0. This category will depend on the choice of a finite totally imaginary Galois extension $E/\mathbb{Q}$ and $\mathfrak{a} \in \mathcal{H}(E/\mathbb{Q})$, so we will sometimes write $\mathrm{RSD}(E, \mathfrak{a}; L)$. However we will also explain how these categories depend on this choice. We will call an object of $\mathrm{RSD}(E, \mathfrak{a}; L)$ a $((E, \mathfrak{a})$-$)$*rational Shimura datum* over a field L. Before turning to the definition we introduce a condition that means that E is sufficiently large to 'see' a particular Shimura variety.

If $G/\mathbb{Q}$ is a reductive group and $\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}}$ and C is a G-conjugacy, defined over a field L of characteristic 0, of cocharacters of G; we will say that a finite Galois extension $E/\mathbb{Q}$ is *acceptable* for (G, ψ, C), if

(1) E is totally imaginary;
(2) G contains a maximal torus T defined over $\mathbb{Q}$ with $T^{\mathrm{ad}}(\mathbb{R})$ compact, which is split by E;
(3) there is a finite set of places S of $\mathbb{Q}$ containing infinity such that

$$\kappa([\psi]) \in \mathbb{Z}[V_{E,S}] \subset \mathbb{Z}[V_E]$$

and $B(\mathbb{Q}, G)_{S,\mathrm{basic}}$ is contained in the image of $H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$;
(4) and, if $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ with $\mathrm{loc}[\phi] = \widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}}\psi_G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi_G})[\psi]$, then ${}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})_E$ has a point in each connected component of ${}^{\phi}G^{\mathrm{ad}}(\mathbb{R})$.

Note that if (G, ψ, C) is any such triple then there is a finite Galois extension $D/\mathbb{Q}$ containing E such that for any $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and any $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D = {}^t \inf_{D/E} \mathfrak{a}$, the field D is acceptable for $(G, \inf^{\mathrm{loc}}_{D/E,t} \psi, C)$. (This follows from point (8) recalled at the end of section 2.3, the finiteness of $\ker^1(\mathbb{Q}, {}^{\phi}G)$ (see theorem 7.1 of [BS]), and the density of ${}^{\phi}G^{\mathrm{ad}}(\mathbb{Q})$ in ${}^{\phi}G^{\mathrm{ad}}(\mathbb{R})$ (see theorem 7.8 of [PR]).) Let $L_0 \subset L$ be the field of definition of C. As C contains an element μ that factors through T, it contains an element that can be defined over E. Thus there is an embedding

$\rho : L_0 \hookrightarrow E$. In this case the $\mathrm{Gal}\,(E/\mathbb{Q})$ orbit of ${}^{\rho}C$ is defined independent of the choice of ρ, and hence $\lambda(C) = \lambda({}^{\rho}C) \in \Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})}$ is well defined.

We now turn to the definition of $\mathrm{RSD}(E, \mathfrak{a}; L)$. An object of $\mathrm{RSD}(E, \mathfrak{a}; L)$ will be a 4-tuple (G, ψ, C, U) where

(1) G is a reductive group over $\mathbb{Q}$;
(2) $\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E))_{\mathrm{basic}}$ with ${}^{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi}G^{\mathrm{ad}}\,(\mathbb{R})$ compact;
(3) C is a G-conjugacy class, defined over L, of miniscule cocharacacters of G such that E is acceptable for (G, ψ, C) and $\overline{\kappa}(\psi) = \lambda(C) \in \Lambda_{G,\mathrm{Gal}\,(E/\mathbb{Q})}$;
(4) and U is an open compact subgroup of ${}^{\psi}G(\mathbb{A}^{\infty})$.

By a *morphism*

$$(\phi, g, f) : (G_1, \psi_1, C_1, U_1) \to (G_2, \psi_2, C_2, U_2)$$

in $\mathrm{RSD}(E, \mathfrak{a}; L)$, we will mean

- a cocycle $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G_2(E))_{\mathrm{basic}}$,
- an element $g \in G_2(\mathbb{A}_E)$,
- a morphism $f : G_1 \to {}^{\phi}G_2$ defined over $\mathbb{Q}$;

such that

- $f \circ \psi_1 = ({}^{g^{-1}}\psi_2)\mathrm{loc}_{\mathfrak{a}}\phi^{-1}$ (so that $\mathrm{conj}_g \circ f : {}^{\psi_1}G_1 \to {}^{\psi_2}G_2$ over $\mathbb{A}$),
- $f(C_1) \subset C_2$,
- and $(\mathrm{conj}_g \circ f)(U_1) \subset U_2$.

We define the composite of such morphisms by

$$(\phi_2, g_2, f_2) \circ (\phi_1, g_1, f_1) = (f_2(\phi_1)\phi_2, g_2 f_2(g_1), f_2 \circ f_1)$$

and set

$$\mathrm{Id}_{(G,\psi,C)} = (1, 1, 1).$$

The purpose of G is simply to provide some base point in a class of extended pure inner forms, and is not very important. If $Z(G)$ is connected, then (by proposition 10.4 of [K2]) any object of $\mathrm{RSD}(E, \mathfrak{a}; L)$ is isomorphic to one with G quasi-split. In this case it would be simpler and more natural to restrict to the full subcategory of 4-tuples (G, ψ, C, U) with G quasi-split, which loses no generality.

If $\tau : L \to L'$ is a map of fields then we define a functor

$$\mathrm{RSD}(\tau) : \mathrm{RSD}(E, \mathfrak{a}; L) \longrightarrow \mathrm{RSD}(E, \mathfrak{a}; L')$$

by

$${}^{\tau}(G, \psi, C, U) = (G, \psi, {}^{\tau}C, U)$$

and

$${}^{\tau}(\phi, g, f) = (\phi, g, f).$$

This functor is fully faithful. Note that

$$\mathrm{RSD}(\tau' \circ \tau) = \mathrm{RSD}(\tau') \circ \mathrm{RSD}(\tau).$$

Suppose that $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D = {}^t \inf_{D/E} \mathfrak{a}$. Then there is a functor

$$\begin{array}{rcl} \inf_{D/E,t} : \mathrm{RSD}(E,\mathfrak{a};L) & \longrightarrow & \mathrm{RSD}(D,\mathfrak{a}_D;L) \\ (G,\psi,C,U) & \longmapsto & (G, \inf_{D/E,t}\psi, C, U) \\ (\phi,g,f) & \longmapsto & (\inf_{D/E,t}\phi, \boldsymbol{\nu}_\phi(t)^{-1}g, f). \end{array}$$

(Note that ${}^\psi G = {}^{\inf_{D/E,t}\psi}G$.) Finally note that $\inf_{D/E,t}$ is faithful and that

$$\inf_{D'/D,t'} \circ \inf_{D/E,t} = \inf_{D'/E,t\eta_{D/E}(t')}.$$

The reader is now in a position to read the statement of the main theorem 4.3, except for the part concerning complex uniformization; and might like to do so. However before stating that theorem we make some auxilliary definitions.

First note that there is an analogue $\mathrm{RSD}(E,\mathfrak{a};L)^-$ of $\mathrm{RSD}(E,\mathfrak{a};L)$ where one suppresses the choice of open compact subgroup U (and the third condition imposed in the definition of a morphism). The functors $\mathrm{RSD}(\tau)$ and $\inf_{D/E,t}$ are still defined in this setting.

We will write $\widetilde{G}_{E,(G,\psi,C)}(\mathbb{A}) = \widetilde{G}_{E,\psi}(\mathbb{A})$ for the subgroup

$$\begin{array}{rl} & \{(\zeta,g,1) \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \times G(\mathbb{A}_E) \times \{1_G\} :\ (\mathrm{loc}_\mathfrak{a}\zeta)^g\psi = \psi\} \\ = & \{(\zeta,g,1) \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_\mathfrak{a}, Z(G)(E))_{\mathrm{basic}} \times G(\mathbb{A}_E) \times \{1_G\} :\ (\mathrm{loc}_\mathfrak{a}\zeta)^g\psi = \psi\} \\ \subset & \mathrm{Aut}_{\mathrm{RSD}(E,\mathfrak{a};L)^-}(G,\psi,C). \end{array}$$

Note that $\widetilde{G}_{E,(G,\psi,C)}(\mathbb{A})$ does not depend on $\mathfrak{a}$ (as the notation suggests). (Nor does it depend on C.) Explicitly we have

$$(\zeta_2,g_2,1)(\zeta_1,g_1,1) = (\zeta_2\zeta_1, g_2g_1, 1).$$

We will often write (ζ,g) for $(\zeta,g,1)$. We have embeddings

$$\begin{array}{rcl} {}^\psi G(\mathbb{A}) & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}) \\ g & \longmapsto & (1,g) \end{array}$$

and

$$\begin{array}{rcl} Z(G)(E) & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}) \\ z & \longmapsto & ({}^z1, z^{-1}). \end{array}$$

We further define

$$\widetilde{G}_{E,\psi}(\mathbb{A}^\infty) = \widetilde{G}_{E,\psi}(\mathbb{A})/\overline{Z(G)(\mathbb{Q})^\psi G(\mathbb{R})}.$$

There is a short exact sequence

$$\begin{array}{l} (0) \longrightarrow {}^\psi G(\mathbb{A}^\infty)/\overline{Z(G)(\mathbb{Q})} \longrightarrow \widetilde{G}_{E,\psi}(\mathbb{A}^\infty) \stackrel{\boldsymbol{\zeta}}{\longrightarrow} \\ \ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E))) \longrightarrow (0), \end{array}$$

where $\boldsymbol{\zeta}$ is induced by $(\zeta,g) \mapsto \zeta$. We endow $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ with a topology by decreeing that ${}^\psi G(\mathbb{A}^\infty)$ is an open subgroup in its usual topology. This makes $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ a totally disconnected, locally compact group. (We stress that the notation is *not* supposed to imply that $\widetilde{G}_{E,(G,\psi,C)}(\mathbb{A})$ and $\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ are the adelic points of any algebraic group.)

Lemma 4.1.

$$\begin{aligned} & \ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E))) \\ = \; & \ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^\psi G(\mathbb{A}_E^\infty)) \oplus H^1(\mathrm{Gal}\,(E/F), Z(G)(E_\infty))). \end{aligned}$$

Proof: This is just the observation that

$$H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C})) \hookrightarrow H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), {}^\psi G(\mathbb{C}))$$

because ${}^\psi G(\mathbb{R}) \twoheadrightarrow {}^\psi G^{\mathrm{ad}}(\mathbb{R})$, because the latter is connected.□

If $(\phi, g, f) : (G_1, \psi_1, C_1) \longrightarrow (G_2, \psi_2, C_2)$, set

$\widetilde{G}_{E,(G_1,\psi_1,C_1)}(\mathbb{A}^\infty)_f = \{(\zeta, h) \in \widetilde{G}_{E,(G_1,\psi_1,C_1)}(\mathbb{A}^\infty) : \; f \circ \zeta \text{ is valued in } Z(G_2)(E)\}.$

Then we get a continuous homomorphism

$$\begin{array}{rcl} \widetilde{\theta}_{(\phi,g,f)} : \widetilde{G}_{E,(G_1,\psi_1,C_1)}(\mathbb{A}^\infty)_f & \longrightarrow & \widetilde{G}_{E,(G_2,\psi_2,C_2)}(\mathbb{A}^\infty) \\ (\zeta, h) & \longmapsto & (f \circ \zeta, \mathrm{conj}_g(f(h))), \end{array}$$

satisfying

$$\widetilde{\theta}_{(\phi,g,f)}|_{{}^{\psi_1}G_1(\mathbb{A}^\infty)/\overline{Z(G_1)(\mathbb{Q})}} = \mathrm{conj}_g \circ f$$

and

$$\widetilde{\theta}_{(\phi,g,f)}(h) \circ (\phi, g, f) = (\phi, g, f) \circ h.$$

We get a map

$$\begin{array}{rcl} \inf_{D/E,t} : \widetilde{G}_{E,(G,\psi,C)}(\mathbb{A}^\infty) & \longrightarrow & \widetilde{G}_{D,\inf_{D/E,t}(G,\psi,C)}(\mathbb{A}^\infty) \\ [(\zeta, g)] & \longmapsto & [(\inf_{\mathrm{Gal}\,(E/\mathbb{Q})}^{\mathrm{Gal}\,(D/\mathbb{Q})} \zeta, g)], \end{array}$$

which only depends on $\mathfrak{a}$ and ${}^t\mathfrak{a}$, but not on t. It restricts to the identity on ${}^\psi G(\mathbb{A}^\infty) = \inf^{\mathrm{loc}}_{D/E,t} {}^\psi G(\mathbb{A}^\infty)$.

4.2. **Labels.** If (G, ψ, C) is a an object of $\mathrm{RSD}(E, \mathfrak{a}; \mathbb{C})^-$, we define $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ to be the set of pairs (ϕ, b) where

(1) $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ with

$$\mathrm{loc}[\phi] = \widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}} {}^\psi G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}} {}^\psi G})[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}},$$

(2) and $b \in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}} \phi = {}^b\mathrm{res}^\infty \psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E^\infty))_{\mathrm{basic}}$.

Note that $\boldsymbol{\nu}_\phi = \boldsymbol{\nu}_\psi \boldsymbol{\nu}_{Y(C)}^{-1}$. We set

$$(G, \psi, C)_{(\phi,b)} = ({}^\phi G, Y(C)_{{}^\phi G}),$$

a Deligne Shimura datum. Note that

$$\mathrm{conj}_b : {}^\psi G(\mathbb{A}^\infty) \xrightarrow{\sim} {}^\phi G(\mathbb{A}^\infty),$$

There is an embedding

$$\begin{array}{rcl} i_{(\phi,b)} : ({}^{\phi}G)(E)^{\mathbb{Q}}_{\mathbb{R}} & \hookrightarrow & \widetilde{G}_{E,\psi}(\mathbb{A}^{\infty}) \\ \gamma & \longmapsto & [((\sigma \mapsto \gamma^{-1}\phi(\sigma)^{\sigma}\gamma\phi(\sigma)^{-1}), (b^{-1}\gamma b, \widehat{\gamma}^{-1}\gamma))], \end{array}$$

where $\widehat{\gamma} \in ({}^{\phi}G)(\mathbb{R})$ lifts $\mathrm{ad}\,\gamma \in ({}^{\phi}G^{\mathrm{ad}})(\mathbb{Q})$. (This is independent of the lift $\widehat{\gamma}$.) Note that

$$i_{(\phi,b)}({}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}}) \cap {}^{\psi}G(\mathbb{A}^{\infty}) = b^{-1\phi}G(\mathbb{Q})b.$$

By lemma 4.1 $\boldsymbol{\zeta}$ gives an isomorphism from $\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})/{}^{\psi}G(\mathbb{A}^{\infty})$ to

$$\ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi}G(\mathbb{A}_E)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))).$$

The image of $i_{(\phi,b)}({}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}})$ is

$$\ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi}G(E)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))).$$

There is an action of $G(E) \times \widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$ on $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ via

$$(\gamma, (\zeta, g))(\phi, b) = ({}^{\gamma}\phi\zeta, \gamma b(g^{\infty})^{-1}).$$

We have

$$i_{(\gamma,\widetilde{g})(\phi,b)} = \mathrm{conj}_{\widetilde{g}} \circ i_{(\phi,b)} \circ \mathrm{conj}_{\gamma}^{-1}.$$

We call two elements of $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ *equivalent* if one is a translate of the other by an element of $G(E) \times ({}^{\psi}G)(\mathbb{A}^{\infty})$. We denote this relations $\sim$. Note that there is a bijection

$$\begin{array}{rcl} \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim & \xrightarrow{\sim} & \{\boldsymbol{\phi} \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E)) :\, \mathrm{loc}\boldsymbol{\phi} = \widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G})[\psi]\} \\ {[(\phi,b)]} & \longmapsto & [\phi]. \end{array}$$

Lemma 4.2. *(1)* $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C) \neq \emptyset$, *and for* $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ *we have*

$$\#(\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim) = \#\ker^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi}G(E)) = \#\ker^1(\mathbb{Q}, {}^{\phi}G),$$

which in particular is finite.

(2) The action of $G(E) \times \widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$ *on* $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ *is transitive.*

(3) The map from

$$\ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi}G(\mathbb{A}_E)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C})))$$

to

$$\ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi}G(\mathbb{A}_E)))$$

is surjective.

(4) The stabilizer in $G(E)\times\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})$ *of* $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ *is* $(1\times i_{(\phi,b)})({}^{\phi}G)(E)^{\mathbb{Q}}_{\mathbb{R}})$.

(5) The stabilizer in $G(E)\times{}^{\psi}G(\mathbb{A}^{\infty})$ *of* $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ *is* $(1\times\mathrm{conj}_b^{-1})({}^{\phi}G(\mathbb{Q}))$.

Proof: For the first part $\overline{\kappa}(\widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}}\psi_G}(Y(C)^{-1})[\psi]) = 0$ and so by proposition 15.1 of [K2] and the definition of acceptable we can find $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$ with $\mathrm{loc}[\phi] = \widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}}\psi_G}(Y(C)^{-1})[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}$. Then there exists $h \in G(\mathbb{A}_E^\infty)$ with $\mathrm{res}^\infty \mathrm{loc}_{\mathfrak{a}}\phi = {}^h\mathrm{res}^\infty\psi \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E^\infty))_{\mathrm{basic}}$.

For the second part it suffices to show that if $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$ both have the image

$$\widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi_G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}\psi_G})[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}},$$

then $\boldsymbol{\phi}_2 = \boldsymbol{\zeta}\boldsymbol{\phi}_1$ for some $\boldsymbol{\zeta} \in H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E))$ with

$$[\psi] = [\psi]\mathrm{loc}(\boldsymbol{\zeta}) \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q}), G(\mathbb{A}_E))_{\mathrm{basic}}.$$

However if $\phi_i \in \boldsymbol{\phi}_i$ then

$$[\phi_2\phi_1^{-1}] \in \ker^1(\mathrm{Gal}\,(E/\mathbb{Q}), {}^{\phi_1}G(E)).$$

By lemma 2.1 $[\phi_2\phi_1^{-1}]$ is the image of some

$$\boldsymbol{\zeta} \in \ker(H^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(\mathbb{A}_E^\infty)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))).$$

Then

- $\boldsymbol{\phi}_2 = \boldsymbol{\zeta}\boldsymbol{\phi}_1 \in H^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q}), G(E))_{\mathrm{basic}}$;
- and $[\psi] = ([\psi]\mathrm{loc}\boldsymbol{\zeta}) \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})^\infty, G(\mathbb{A}_E))_{\mathrm{basic}}$.

This establishes the second assertion.

The third assertion follows from lemma 2.1 and the definition of acceptable.

For the fourth assertion suppose that $\gamma \in G(E)$ and $(\zeta, g) \in \widetilde{G}_{E,\psi}(\mathbb{A})$ with

$$({}^\gamma\phi\zeta, \gamma b(g^\infty)^{-1}) = (\phi, b).$$

Then

- $\zeta(\sigma) = \phi(\sigma)^\sigma\gamma\phi(\sigma)^{-1}\gamma^{-1} = \gamma^{-1}\phi(\sigma)^\sigma\gamma\phi(\sigma)^{-1}$ and so $\gamma \in {}^\phi G(E)^{\mathbb{Q}}$;
- $g^\infty = b^{-1}\gamma b$;
- if w is an infinite place of E, then $\mathrm{res}_{E_w/\mathbb{R}}\mathrm{loc}[\zeta] \in H^1_{\mathrm{alg}}(W_{E_w/\mathbb{R}}, {}^\psi G(E_w))_{\mathrm{basic}}$ is trivial, and so $\mathrm{res}_{E_w/\mathbb{R}}\mathrm{loc}[\zeta] \in H^1(\mathrm{Gal}\,(E_w/\mathbb{R}), Z(G)(E_w))$ is trivial (because ${}^\psi G^{\mathrm{ad}}(\mathbb{R})$ is connected);
- $\gamma \in {}^\phi G(E)^{\mathbb{Q}}_{\mathbb{R}}$ and $(\zeta, g) = i_{(\phi,b)}(\gamma)h$ with $h \in G(E_\infty)$;
- ${}^h\mathrm{res}_\infty\psi = \mathrm{res}_\infty\psi$ and so $h \in {}^\psi G(\mathbb{R})$.

This establishes the fourth part. The final part follows from this. □

If $(\phi, g, f) : (G_1, \psi_1, C_1) \to (G_2, \psi_2, C_2)$ then we get a map

$$\begin{array}{rcl} \mathrm{Label}_{\mathfrak{a}}(\phi, g, f) : \mathrm{Label}_{\mathfrak{a}}(G_1, \psi_1, C_1) & \longrightarrow & \mathrm{Label}_{\mathfrak{a}}(G_2, \psi_2, C_2) \\ (\phi_1, b_1) & \longmapsto & ((f \circ \phi_1)\phi, f(b_1)g^{-1}). \end{array}$$

To see that this is well defined, note that

$$\boldsymbol{\nu}_{f\circ\phi_1} = (f \circ \boldsymbol{\nu}_{\psi_1})(f \circ \boldsymbol{\nu}_{Y(C_1)})^{-1} = \boldsymbol{\nu}_{\psi_2}\boldsymbol{\nu}_\phi^{-1}\boldsymbol{\nu}_{Y(C_2)}^{-1}$$

factors through $Z(G_2)$ and so $(f \circ \phi_1)\phi$ is basic. Moreover f gives a map

$$f : {}^{\phi_1}G_1 \longrightarrow {}^{(f\circ\phi_1)\phi}G_2$$

over $\mathbb{Q}$ which takes $Y(C_1)_{{}^{\phi_1}G_1}$ to $Y(C_2)_{{}^{(f\circ\phi_1)\phi}G_2}$, i.e.

$$f : ({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1}) \longrightarrow ({}^{(f\circ\phi_1)\phi}G_2, Y(C_2)_{{}^{(f\circ\phi_1)\phi}G_2}).$$

Moreover

$$\mathrm{conj}_{f(b_1)g^{-1}} \circ \mathrm{conj}_g \circ f = f \circ \mathrm{conj}_{b_1} : {}^{\psi_1}G_1(\mathbb{A}^\infty) \longrightarrow {}^{(f\circ\phi_1)\phi}G_2(\mathbb{A}^\infty).$$

Additionally, if $(\gamma, h) \in G(E) \times {}^{\psi}G(\mathbb{A}^\infty)$, then

$$\mathrm{Label}_{\mathfrak{a}}(\phi, g, f)({}^{(\gamma,h)}(\phi_1, b_1)) = {}^{(f(\gamma), gf(h)g^{-1})}\mathrm{Label}_{\mathfrak{a}}(\phi, g, f)(\phi_1, b_1),$$

and so we get an induced map

$$\mathrm{Label}_{\mathfrak{a}}(\phi, g, f) : (\mathrm{Label}_{\mathfrak{a}}(G_1, \psi_1, C_1)/\sim) \longrightarrow (\mathrm{Label}_{\mathfrak{a}}(G_2, \psi_2, C_2)/\sim).$$

If $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$ and $(\tau, \phi', h) \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi}G, Y(C)_{{}^{\phi}G})$, then

$$(\phi'\phi, hb) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, {}^{\tau}C).$$

(To verify this use the fact from the end of section 2.4 that

$$\begin{aligned} &\widehat{\boldsymbol{\lambda}}_{\widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G})({}^{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G})}({}^{\tau}C - Y(C)_{\widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G})({}^{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G})})\widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}(Y(C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}) \\ = \;& \widehat{\boldsymbol{\lambda}}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}(Y({}^{\tau}C)^{-1}_{\mathrm{res}_{\mathbb{C}/\mathbb{R}}{}^{\psi}G}).) \end{aligned}$$

If $D \supset E$ is another finite Galois extension of $\mathbb{Q}$, that $\mathfrak{a}_D \in \mathcal{H}(D/\mathbb{Q})$ and that $t \in T_{2,E}(\mathbb{A}_D)$ with $\eta_{D/E,*}\mathfrak{a}_D = {}^{t}\inf_{D/E}\mathfrak{a}$. Then there is a map

$$\begin{array}{rcl} \inf_{D/E,t} : \mathrm{Label}_{\mathfrak{a}}(G, \psi, C) & \longrightarrow & \mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(G, \psi, C)) \\ (\phi, b) & \longmapsto & (\inf_{D/E,t}\phi, \boldsymbol{\nu}_{\phi}(t)b). \end{array}$$

It induces a bijection

$$(\mathrm{Label}_{\mathfrak{a}}(G, \psi, C)/\sim) \xrightarrow{\sim} (\mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(G, \psi, C))/\sim).$$

(Because, if $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, then $\ker^1(\mathrm{Gal}(E/\mathbb{Q}), {}^{\phi}G(E)) \xrightarrow{\sim} \ker^1(\mathrm{Gal}(D/\mathbb{Q}), {}^{\phi}G(D))$, as E is acceptable for G.) We have

$$\inf_{D/E,t} \circ \mathrm{Label}_{\mathfrak{a}}(\phi, g, f) = \mathrm{Label}_{\mathfrak{a}_D}(\inf_{D/E,t}(\phi, g, f)) \circ \inf_{D/E,t}$$

and

$$\inf_{D'/D,t'} \circ \inf_{D/E,t} = \inf_{D'/E,t\eta_{D/E}(t')}.$$

4.3. **Rational Shimura varieties.** We now state and prove our second main theorem.

Theorem 4.3. *We have the following objects:*

(I) For any finite totally imaginary Galois extension $E/\mathbb{Q}$*, any* $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$*, and any field* L *of characteristic* 0*; we may associate a functor*

$$\mathrm{Sh}_{E,\mathfrak{a}^+;L} = \mathrm{Sh} : \mathrm{RSD}(E, \mathfrak{a}; L) \longrightarrow \mathrm{QProj}_L.$$

(II) To an embedding of fields $\tau : L \hookrightarrow L'$ *we may associate a natural isomorphism*

$$\Phi_{E,\mathfrak{a}^+}(\tau) = \Phi(\tau) : \tau \circ \mathrm{Sh}_{E,\mathfrak{a}^+;L} \xrightarrow{\sim} \mathrm{Sh}_{E,\mathfrak{a}^+;L'} \circ \mathrm{RSD}(\tau).$$

(III) If $D \supset E$ *is another finite Galois extension of* $\mathbb{Q}$*, if* $\mathfrak{a}_D^+ \in \mathcal{H}(D/\mathbb{Q})^+$ *and if* $t \in T_{2,E}(\mathbb{A}_D)$ *with* $\eta_{D/E,*}\mathfrak{a}_D^+ = {}^t \inf_{D/E} \mathfrak{a}^+$*, there is a natural isomorphism*

$$\alpha_t : \mathrm{Sh}_{E,\mathfrak{a}^+;L} \xrightarrow{\sim} \mathrm{Sh}_{D,\mathfrak{a}_D^+,L} \circ \inf_{D/E,t}.$$

(IV) If (G, ψ, C, U) *is an object of* $\mathrm{RSD}(E, \mathfrak{a}; \mathbb{C})$ *with* U *sufficiently small and if* $(\phi, h) \in \mathrm{Label}_{E,\mathfrak{a}}(G, \psi, C)$*, then* $\mathrm{Sh}_{E,\mathfrak{a}^+;\mathbb{C}}(G, \psi, C, U)$ *is smooth and we may associate an isomorphism of complex manifolds*

$$\pi_{E,\mathfrak{a}^+;(\phi,h)} = \pi_{(\phi,h)} : {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{{}^{\phi}G}) \xrightarrow{\sim} \mathrm{Sh}_{E,\mathfrak{a}^+;\mathbb{C}}(G, \psi, C, U)(\mathbb{C}).$$

These objects satisfy the following properties.

(1) $\mathrm{Sh}(G, \psi, C, U)$ *is normal, and if* U *is sufficiently small, then it is smooth.*

(2) If U *is sufficiently small then the group of automorphisms of the variety* $\mathrm{Sh}(G, \psi, C, U)$ *is finite.*

(3) If $\phi \in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}}, G(E))_{\mathrm{basic}}$*, then*

$$\mathrm{Sh}(\phi, 1, 1) : \mathrm{Sh}({}^{\phi}G, \psi \mathrm{loc}_{\mathfrak{a}} \phi^{-1}, C, U) \xrightarrow{\sim} \mathrm{Sh}(G, \psi, C, U).$$

(This expresses the independence of the choice of 'base point' G*.)*

(4) If $z \in Z(G)(E)$ *and* $u \in U$ *and* $h \in {}^{\psi}G(\mathbb{R})$*, then*

$$\mathrm{Sh}({}^{z}1, z^{-1}uh, 1) : \mathrm{Sh}(G, \psi, C, U) \longrightarrow \mathrm{Sh}(G, \psi, C, U)$$

is the identity. In particular $\widetilde{G}_\psi(\mathbb{A}^\infty)$ *acts on the inverse system* $\{\mathrm{Sh}(G, \psi, C)_U\}_U$*.*

(5) $\mathrm{Sh}(1, g, 1) : \mathrm{Sh}(G, \psi, C, U) \to \mathrm{Sh}(G, \psi, C, V)$ *is a faithfully flat, finite morphism of degree* $[V\overline{Z(G)(\mathbb{Q})} : gUg^{-1}\overline{Z(G)(\mathbb{Q})}]$*. If* V *is sufficiently small then it is etale.*

(6) If $U \lhd V$*, then*

$$\mathrm{Sh}(1, 1, 1) : \mathrm{Sh}(G, \psi, C, U) \longrightarrow \mathrm{Sh}(G, \psi, C, V)$$

is Galois with group $V\overline{Z(G)(\mathbb{Q})}/U\overline{Z(G)(\mathbb{Q})} \cong V/U(\overline{Z(G)(\mathbb{Q})} \cap V)$ *acting via* $v \mapsto \mathrm{Sh}(1, v, 1)$*.*

(7) $\Phi(1) = \mathrm{Id}$ *and* $\Phi(\tau' \circ \tau) = \Phi(\tau') \circ {}^{\tau'}\Phi(\tau)$*.*

(8) $\alpha_t \circ \Phi_{E,\mathfrak{a}^+}(\tau) = \Phi_{D,\mathfrak{a}_D^+}(\tau) \circ \alpha_t$*.*

(9) *If $D' \supset D$ is another finite Galois extension of $\mathbb{Q}$, if $\mathfrak{a}^+_{D'} \in \mathcal{H}(D'/\mathbb{Q})^+$ and if $t' \in T_{2,D}(\mathbb{A}_{D'})$ with $\eta_{D'/D,*}\mathfrak{a}^+_{D'} = {}^{t'}\inf_{D'/D}\mathfrak{a}^+_D$; then $\eta_{D'/E,*}\mathfrak{a}^+_{D'} = {}^{t\eta_{D/E}(t')}\inf_{D'/E}\mathfrak{a}^+$ and*

$$\alpha_{t\eta_{D/E}(t')} = \alpha_{t'} \circ \alpha_t.$$

(10) *If $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$, then $\mathrm{Sh}(\widetilde{g}_1) \circ \pi_{(\phi,b)}([(\widetilde{g}_2 U, \mu)]) = \pi_{(\phi,b)}([(\widetilde{g}_2\widetilde{g}_1^{-1}(\widetilde{g}_1 U \widetilde{g}_1^{-1}), \mu)])$.*
(11) $\pi_{({}^\gamma\phi, \gamma b h^{-1})}([(h\widetilde{g}U, \mathrm{conj}_\gamma(\mu))]) = \pi_{(\phi,b)}([(\widetilde{g}U, \mu)])$.
(12) *If $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$, then $\pi_{\widetilde{g}_1(\phi,b)}([(\widetilde{g}_1\widetilde{g}_2 U, \mu)]) = \pi_{(\phi,b)}([(\widetilde{g}_2 U, \mu)])$.*
(13) $\mathrm{Sh}(\phi, g, f) \circ \pi_{(\phi_1,b_1)}([(\zeta, k)], \mu) = \pi_{(f(\zeta^{-1}\phi_1)\phi, f(b_1 k)g^{-1})}(1, f\circ\mu)$. *In particular, if $\widehat{k} \in \widehat{G}_{1,\psi_1}(\mathbb{A}^\infty)_f$, then*

$$\mathrm{Sh}(\phi, g, f) \circ \pi_{\rho,(\phi_1,b_1)}(\widehat{k}, \mu) = \pi_{\mathrm{Label}_{\mathfrak{a}}(\phi,g,f)(\phi_1,b_1)}(\widetilde{\theta}_{(\phi,g,f)}(\widehat{k}), f\circ\mu).$$

(14) *If $G = T$ is a torus and $\tau \in \mathrm{Aut}(\mathbb{C})$, then*

$$\Phi(\tau)(\tau \circ \pi_{(\phi,b)})(\widetilde{g}, \mu) = \pi_{(\phi_\tau\phi, b_\tau b)}(\widetilde{g}, {}^\tau\mu),$$

for any $(\tau, \phi_\tau, b_\tau) \in \mathrm{Conj}_{\mathfrak{a}}(T, \{\mu\})$ for which b_τ lifts $\overline{b}_{\mathfrak{a}^+,\infty,\mu,\tau} \in T(\mathbb{A}_E^\infty)/\overline{T(\mathbb{Q})}T(E)$. Such a pair (ϕ_τ, b_τ) always exists.
(15) $\alpha_t \circ \pi_{E,\mathfrak{a}^+;\rho,(\phi,b)} = \pi_{D,\mathfrak{a}^+_D;\rho,\inf_{D/E,t}(\phi,b)} \circ (\inf_{D/E,t} \times 1)$.

Before proving this theorem we will give an example of how one can make use of the 'density of special points' in this optic. The argument is the usual one, but rephrased in our language. The particular corollary we prove is rather technical, but our reason to prove it here as an example of this sort of argument and will be needed in section 4.5.

Corollary 4.4. *The set of points in $\mathrm{Sh}(G, \psi, C, U)(\overline{L})$ whose orbit under $\Phi(\tau)\circ\tau$ for $\tau \in \mathrm{Stab}_{\mathrm{Aut}(\overline{L})}(C)$ is finite are Zariski dense. If $L = \mathbb{C}$, then these points are even dense in the Archimedean topology.*

Proof: Let S denote a finite set of places of $\mathbb{Q}$ containg ∞ and every place v with $[\mathrm{res}_v\psi]$ is non-trivial and a finite place v_0 at which G splits and $[\mathrm{res}_{v_0}\psi]$ is trivial. For $v \in S$ choose a maximal torus $T_v \subset G$ defined over $\mathbb{Q}_v$ such that

- $[\mathrm{res}_v\psi_v]$ is in the image of $B(\mathbb{Q}_v, T_v)_{G-\mathrm{basic}}$ for all $v \in S$,
- T_{v_0} is split,
- and $T^{\mathrm{ad}}_\infty(\mathbb{R})$ is compact.

Then choose a torus $T \subset G$ defined over $\mathbb{Q}$ with T conjugate over $\mathbb{Q}_v$ to T_v for all $v \in S$. (See corollary 3 to proposition 7.3 of [PR].) Write i for the inclusion $T \hookrightarrow G$. We can choose $\mu \in C$ which factors through T. We can also choose a finite Galois extension $D/\mathbb{Q}$ containing E which splits T, and $\mathfrak{a}^+_D \in \mathcal{H}(D/\mathbb{Q})^+$, and $t \in T_{2,E}(\mathbb{A}_D)$ with ${}^t\inf_{D/E}\mathfrak{a}^+ = \eta_{D/E,*}\mathfrak{a}^+_D$, and $\psi_T \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(D/\mathbb{Q})_{\mathfrak{a}_D}, T(\mathbb{A}_D))_{G-\mathrm{basic}}$ with $[\psi_T] = \inf_{D/E}[\psi] \in H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(D/\mathbb{Q}), G(\mathbb{A}_D))_{\mathrm{basic}}$. Altering ψ_T only at v_0 we may alter $\overline{\kappa}(\psi_T) \in X_*(T)_{\mathrm{Gal}\,(D/\mathbb{Q})}$ by any element in the image of $X_*(T^{\mathrm{sc}})$. As $\overline{\kappa}(\psi_T)$ maps to $\lambda(C)$ in $(X_*(T)/X_*(T^{\mathrm{sc}}))_{\mathrm{Gal}\,(D/\mathbb{Q})}$ and as

$$X_*(T^{\mathrm{sc}})_{\mathrm{Gal}\,(D/\mathbb{Q})} \longrightarrow X_*(T)_{\mathrm{Gal}\,(D/\mathbb{Q})} \longrightarrow (X_*(T)/X_*(T^{\mathrm{sc}}))_{\mathrm{Gal}\,(D/\mathbb{Q})} \longrightarrow (0)$$

is right exact, we may arrange that $\lambda(\{\mu\}) = \overline{\kappa}(\psi_T) \in X_*(T)_{\mathrm{Gal}\,(D/\mathbb{Q})}$. Next choose $g \in G(\mathbb{A}_D)$ with ${}^g\psi_T = \inf_{D/E,t}\psi$.

If $\widetilde{k} \in \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$ and $V \subset T(\mathbb{A}^\infty)$ with $\widetilde{k}gVg^{-1}\widetilde{k}^{-1} \subset U$ then

$$\mathrm{Sh}(\widetilde{k}) \circ \alpha_t^{-1} \circ \mathrm{Sh}(1,g,i) : \mathrm{Sh}_{D,\mathfrak{a}_D^+}(T,\psi_T,\{\mu\},V) \longrightarrow \mathrm{Sh}_{E,\mathfrak{a}}(G,\psi,C,U).$$

The image is finite and is preserved by $\Phi(\tau)$ for $\tau \in \mathrm{Aut}\,(L)$ which fixes C. As $\widetilde{k}$ and V vary, the union of the images is Zariski dense in $\mathrm{Sh}_{E,\mathfrak{a}}(G,\psi,C,U)$. If $L = \mathbb{C}$ it is even dense in the Archimedean topology. □

One can use a similar argument to prove the uniqueness up to unique isomorphism of objects satisfying the theorem. First of all one considers the case $L = \mathbb{C}$. In this case uniqueness of everything except the $\Phi(\tau)$ is clear. The $\Phi(\tau)$ are unique in the case that G is a torus. Then an argument as in the corollary shows they are unique for all G. Once one knows the uniqueness for $L = \mathbb{C}$ one can deduce it for L a number field, and then for L any field of characteristic 0.

We will prove the theorem over the next two sections. We will first treat the case $L = \mathbb{C}$. The general case will follow from this rather formally. (Note that once we prove the theorem in the case $L = \mathbb{C}$, the corollary follows in that case.)

4.4. **Proof in the case $L = \mathbb{C}$.** If $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ we define

$$\mathrm{Sh}(G,\psi,C,U)_{(\phi,b)} = \mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{bUb^{-1}}.$$

Up to canonical isomorphism this only depends on the equivalence class of (ϕ, b). Indeed if $\gamma \in G(E)$ and $h \in {}^\psi G(\mathbb{A}^\infty)$, then $\mathrm{conj}_\gamma : {}^\phi G \xrightarrow{\sim} {}^{\gamma\phi}G$ and

$$\mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1}, \mathrm{conj}_\gamma) : \mathrm{Sh}(G,\psi,C,U)_{(\phi,b)} \xrightarrow{\sim} \mathrm{Sh}(G,\psi,C,U)_{({}^\gamma\phi,\gamma bh^{-1})},$$

i.e.

$$\mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1}, \mathrm{conj}_\gamma) : ({}^\phi G, Y(C)_{{}^\phi G})_{bUb^{-1}} \xrightarrow{\sim} ({}^{\gamma\phi}G, Y(C)_{{}^{\gamma\phi}G})_{\gamma bh^{-1}U(\gamma bh^{-1})^{-1}}.$$

If we replace (γ, h) by $(\gamma\delta, h\mathrm{conj}_{b^{-1}}(\delta))$ with $\delta \in {}^\phi G(\mathbb{Q})$ then this map is replaced by

$$\begin{aligned}\mathrm{Sh}(\mathrm{conj}_{\gamma\delta b}(hb^{-1}\delta b)^{-1}, \mathrm{conj}_{\gamma\delta}) &= \mathrm{Sh}(\gamma bh^{-1}b^{-1}\delta^{-1}\gamma^{-1}, \mathrm{conj}_{\gamma\delta}) \\ &= \mathrm{Sh}(\gamma bh^{-1}b^{-1}\gamma^{-1}, \mathrm{conj}_\gamma) \circ \mathrm{Sh}(\delta^{-1}, \mathrm{conj}_\delta) \\ &= \mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1}, \mathrm{conj}_\gamma).\end{aligned}$$

Thus we have a canonical isomorphism

$$\alpha_{(\phi,b),({}^\gamma\phi,\gamma bh^{-1})} : \mathrm{Sh}(G,\psi,C,U)_{(\phi,b)} \xrightarrow{\sim} \mathrm{Sh}(G,\psi,C,U)_{({}^\gamma\phi,\gamma bh^{-1})}$$

which is independent of the choice of (γ, h), and $\mathrm{Sh}(G,\psi,C,U)_{(\phi,b)}$ only depends on $[(\phi,b)]$ up to canonical isomorphism. Thus we can define

$$\mathrm{Sh}(G,\psi,C,U) = \coprod_{[(\phi,b)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \mathrm{Sh}(G,\psi,C,U)_{(\phi,b)}$$

and it is well defined up to canonical isomorphism. As the union is finite, $\mathrm{Sh}(G,\psi,C,U)$ is a quasi-projective variety. It is normal, and if U is sufficiently small then it is smooth and its group of automorphisms is finite. (As bUb^{-1} is again sufficiently small.)

Now suppose that $(\phi, g, f) : (G_1,\psi_1,C_1,U_1) \to (G_2,\psi_2,C_2,U_2)$. Then we define

$$\mathrm{Sh}(\phi,g,f)|_{\mathrm{Sh}(G_1,\psi_1,C_1,U_1)_{(\phi_1,b_1)}} : \mathrm{Sh}(G_1,\psi_1,C_1,U_1)_{(\phi_1,b_1)} \longrightarrow \mathrm{Sh}(G_2,\psi_2,C_2,U_2)_{((f\circ\phi_1)\phi,f(b_1)g^{-1})}$$

to be

$$\mathrm{Sh}(1,f) : \mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1U_1b_1^{-1}} \longrightarrow \mathrm{Sh}({}^{(f\circ\phi_1)\phi}G_2, Y(C_2)_{{}^{(f\circ\phi_1)\phi}G_2})_{f(b_1)g^{-1}U_2gf(b_1)^{-1}}.$$

This is well defined independent of the choice of representatives (ϕ_1,b_1) because, for $\gamma\in G(E)$ and $h \in {}^{\psi_1}G_1(\mathbb{A}^\infty)$, we have

$$\mathrm{Label}_{\mathfrak{a}}(\phi,g,f)({}^{(\gamma,h)}(\phi_1,b_1)) = {}^{(f(\gamma),gf(h)g^{-1})}\mathrm{Label}_{\mathfrak{a}}(\phi,g,f)(\phi_1,b_1)$$

and

$$\mathrm{Sh}(1,f)\circ\mathrm{Sh}(\mathrm{conj}_{\gamma b_1}(h)^{-1},\mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{f(\gamma)f(b_1)g^{-1}}(gf(h)g^{-1})^{-1},\mathrm{conj}_{f(\gamma)})\circ\mathrm{Sh}(1,f).$$

We have $\mathrm{Sh}(1,1,1)=\mathrm{Id}$ and $\mathrm{Sh}((\phi',g',f')\circ(\phi,g,f)) = \mathrm{Sh}(\phi',g',f')\circ\mathrm{Sh}(\phi,g,f)$ (where $(\phi',g',f') : (G_2,\psi_2,C_2,U_2)\to(G_3,\psi_3,C_3,U_3)$). To verify the latter suppose that $(\phi_1,b_1)\in\mathrm{Label}_{\mathfrak{a}}(G_1,\psi_1,C_1)$. Then we have to verify that $\mathrm{Sh}(1,f_2)\circ\mathrm{Sh}(1,f_1) = \mathrm{Sh}(1,f_1f_2)$ as maps

$$\begin{array}{rcl} \mathrm{Sh}(G_1,\psi_1,C_1,U_1)_{(\phi_1,b_1)} & \longrightarrow & \mathrm{Sh}(G_3,\psi_3,C_3)_{U_3,((f_2\circ((f_1\circ\phi_1)\phi))\phi',f_2(f_1(b_1)g_1^{-1})g_2^{-1})} \\ & = & \mathrm{Sh}(G_3,\psi_3,C_3)_{U_3,((f_2\circ f_1\circ\phi_1)((f_2\circ\phi)\phi'),(f_2\circ f_1)(b_1)(g_2f(g_1))^{-1})} \end{array},$$

which is clear.

If $\phi\in Z^1_{\mathrm{alg}}(\mathcal{E}_3(E/\mathbb{Q})_{\mathfrak{a}},G(E))_{\mathrm{basic}}$, then

$$\mathrm{Sh}(\phi,1,1) : \mathrm{Sh}({}^{\phi}G, \psi\mathrm{loc}_{\mathfrak{a}}\phi^{-1},C,U) \longrightarrow \mathrm{Sh}(G,\psi,C,U)$$

has two-sided inverse $\mathrm{Sh}(\phi^{-1},1,1)$ and so is an isomorphism.

If $z\in Z(G)(E)$ and $u\in U$ and $h\in{}^{\psi}G(\mathbb{R})$ then we must show that

$$\mathrm{Sh}({}^{z}1, z^{-1}uh, 1) : \mathrm{Sh}(G,\psi,C,U) \longrightarrow \mathrm{Sh}(G,\psi,C,U)$$

is the identity. First note that it is an isomorphism, because it has two-sided inverse $\mathrm{Sh}({}^{z^{-1}}1, zu^{-1}h^{-1},1)$. Moreover its restriction to $\mathrm{Sh}(G,\psi,C,U)_{(\phi_1,b_1)}$ is

$$\mathrm{Sh}(1,1) : \mathrm{Sh}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})_{b_1Ub_1^{-1}} \longrightarrow \mathrm{Sh}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})_{b_1Ub_1^{-1}} = \mathrm{Sh}(G,\psi,C)_{U,(\phi_1{}^{z}1,b_1u^{-1}z)}.$$

Thus it suffices to check that

$$\alpha_{(\phi_1,b_1),({}^{z}\phi_1,b_1u^{-1}z)} = \mathrm{Sh}(\mathrm{conj}_{zb_1}(u^{-1}),\mathrm{conj}_z) = \mathrm{Sh}(b_1u^{-1}b_1^{-1},1)$$

equals $\mathrm{Sh}(1,1)$ on $\mathrm{Sh}({}^{\phi_1}G, Y(C)_{\phi_1 G})_{b_1 U b_1^{-1}}$, which it does.

The map

$$\mathrm{Sh}(1,g,1) : \mathrm{Sh}(G,\psi,C,U) \to \mathrm{Sh}(G,\psi,C,V)$$

equals

$$\coprod_{[(\phi_1,b_1)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \mathrm{Sh}(1,1) : \mathrm{Sh}(G,\psi,C,U)_{(\phi_1,b_1)} \longrightarrow \mathrm{Sh}(G,\psi,C,V)_{(\phi_1,b_1g^{-1})},$$

i.e.

$$\coprod_{[(\phi_1,b_1)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} (\alpha_{(\phi_1,b_1g^{-1}),(\phi_1,b_1)} \circ \mathrm{Sh}(1,1)) : \mathrm{Sh}(G,\psi,C,U)_{(\phi_1,b_1)} \longrightarrow \mathrm{Sh}(G,\psi,C,V)_{(\phi_1,b_1)},$$

i.e.

$$\coprod_{[(\phi_1,b_1)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} (\mathrm{Sh}(\mathrm{conj}_{b_1}(g),1) : \mathrm{Sh}({}^{\phi_1}G, Y(C)_{\phi_1 G})_{b_1 U b_1^{-1}} \longrightarrow \mathrm{Sh}({}^{\phi_1}G, Y(C)_{\phi_1 G})_{b_1 V b_1^{-1}}.$$

This is finite and faithfully flat of degree

$$[b_1 V b_1^{-1} Z(G)(\mathbb{Q}) : b_1 g U g^{-1} b_1^{-1} Z(G)(\mathbb{Q})] = [V\overline{Z(G)(\mathbb{Q})} : gUg^{-1}\overline{Z(G)(\mathbb{Q})}].$$

If V is sufficiently small, then so is $b_1 V b_1^{-1}$ and so this map is etale. If moreover $U \lhd V$, then we also see that

$$\mathrm{Sh}(1,1,1) : \mathrm{Sh}(G,\psi,C,U) \longrightarrow \mathrm{Sh}(G,\psi,C,V)$$

is Galois with group $V\overline{Z(G)(\mathbb{Q})}/U\overline{Z(G)(\mathbb{Q})}$ acting via $v \mapsto \mathrm{Sh}(1,v,1)$.

Next suppose that $\tau \in \mathrm{Aut}\,(\mathbb{C})$. We define

$$\begin{aligned} & (\Phi(\tau) : {}^{\tau}\mathrm{Sh}(G,\psi,C,U) \longrightarrow \mathrm{Sh}(G,\psi,{}^{\tau}C,U)) \\ = \ & \textstyle\coprod_{[(\phi_1,b_1)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \left(\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) : {}^{\tau}\mathrm{Sh}(G,\psi,C,U)_{(\phi_1,b_1)} \xrightarrow{\sim} \mathrm{Sh}(G,\psi,{}^{\tau}C,U)_{(\phi\phi_1,bb_1)} \right) \\ = \ & \textstyle\coprod_{[(\phi_1,b_1)]\in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim} \left(\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) : {}^{\tau}\mathrm{Sh}({}^{\phi_1}G, Y(C)_{\phi_1 G})_{b_1 U b_1^{-1}} \xrightarrow{\sim} \mathrm{Sh}({}^{\phi\phi_1}G, Y({}^{\tau}C)_{\phi\phi_1 G})_{bb_1 U b_1^{-1} b^{-1}} \right) \end{aligned}$$

for any $(\tau,\phi,b) \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi_1}G, Y(C)_{\phi_1 G})$. Note that ${}^{\tau,\phi}Y(C)_{\phi_1 G} = Y({}^{\tau}C)_{\phi\phi_1 G}$. To see that this is independent of the choice of $(\tau,\phi,b) \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi_1}G, Y(C)_{\phi_1 G})$ we must check that if $\gamma \in G(E)$ and $h \in {}^{\phi_1}G(\mathbb{A}^{\infty})$, then

$$\alpha_{(\phi\phi_1,bb_1),(({}^{\gamma}\phi)\phi_1,\gamma b h^{-1} b_1)} \circ \Phi(\tau,\phi,b) = \Phi(\tau,{}^{\gamma}\phi,\gamma b h^{-1}),$$

i.e. that

$$\mathrm{Sh}(\mathrm{conj}_{\gamma b b_1}(b_1^{-1}h^{-1}b_1), \mathrm{conj}_{\gamma}) \circ \Phi(\tau,\phi,b) = \Phi(\tau,{}^{\gamma}\phi,\gamma b h^{-1}).$$

However both sides equals $\mathrm{Sh}(1,\mathrm{conj}_{\gamma}) \circ \Phi(\tau,\phi,b) \circ {}^{\tau}\mathrm{Sh}(h^{-1},1)$. It is also independent of the choice of representatives (ϕ_1,b_1) because, if $\gamma \in G(E)$ and $h \in {}^{\psi}G(\mathbb{A}^{\infty})$ then $(\tau, \mathrm{conj}_{\gamma} \circ \phi, \mathrm{conj}_{\gamma}(b)) \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\gamma\phi_1}G, Y(C)_{\gamma\phi_1 G})$ and

$$\Phi(\tau,\mathrm{conj}_{\gamma}\circ\phi,\mathrm{conj}_{\gamma}(b)) \circ {}^{\tau}\alpha_{(\phi_1,b_1),({}^{\gamma}\phi_1,\gamma b_1 h^{-1})} = \alpha_{(\phi\phi_1,bb_1),((\mathrm{conj}_{\gamma}\circ\phi){}^{\gamma}\phi_1,\gamma b b_1 h^{-1})} \circ \Phi(\tau,\phi,b).$$

To see this, note that $(\mathrm{conj}_\gamma \circ \phi)^\gamma \phi_1 = {}^\gamma(\phi\phi_1)$ and decode the equality to

$$\Phi(\tau, \mathrm{conj}_\gamma \phi, \gamma b \gamma^{-1}) \circ {}^\tau\mathrm{Sh}(\mathrm{conj}_{\gamma b_1}(h^{-1}), \mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{\gamma b b_1}(h^{-1}), \mathrm{conj}_\gamma) \circ \Phi(\tau, \phi, b),$$

which holds because

$$\begin{array}{rl} & \Phi(\tau, \mathrm{conj}_\gamma \phi, \gamma b \gamma^{-1}) \circ {}^\tau\mathrm{Sh}(\mathrm{conj}_{\gamma b_1}(h^{-1}), \mathrm{conj}_\gamma) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\gamma) \circ \Phi(\tau, \phi, b) \circ {}^\tau\mathrm{Sh}(\mathrm{conj}_{b_1}(h^{-1}), 1) \\ = & \mathrm{Sh}(1, \mathrm{conj}_\gamma) \circ \mathrm{Sh}(\mathrm{conj}_{b b_1}(h^{-1}), 1) \circ \Phi(\tau, \phi, b) \\ = & \mathrm{Sh}(\mathrm{conj}_{\gamma b b_1}(h^{-1}), \mathrm{conj}_\gamma) \circ \Phi(\tau, \phi, b). \end{array}$$

We have $\Phi(1) = \mathrm{Id}$ and $\Phi(\tau'\tau) = \Phi(\tau') \circ {}^{\tau'}\Phi(\tau)$. The latter because if $(\tau, \phi, b) \in \mathrm{Conj}\,({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})$ and $(\tau', \phi', b') \in \mathrm{Conj}\,({}^{\phi\phi_1}G, Y({}^\tau C)_{{}^{\phi\phi_1}G})$, then $(\tau'\tau, \phi'\phi, b'b) \in \mathrm{Conj}\,({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})$ and

$$\Phi(\tau'\tau, \phi'\phi, b'b) = \Phi(\tau', \phi', b') \circ {}^{\tau'}\Phi(\tau, \phi, b).$$

We must check that $\Phi(\tau) \circ {}^\tau\mathrm{Sh}(\phi, g, f) = \mathrm{Sh}(\phi, g, f) \circ \Phi(\tau)$. Consider the restriction of both sides to

$${}^\tau\mathrm{Sh}(G_1, \psi_1, C_1, U_1)_{(\phi_1, b_1)} = {}^\tau\mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1 U_1 b_1^{-1}}.$$

Choose $(\phi', b') \in \mathrm{Conj}\,({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})$, so that $(f \circ \phi', f(b')) \in \mathrm{Conj}\,({}^{(f\circ\phi_1)\phi}G_2, Y(C_2)_{{}^{(f\circ\phi_1)\phi}G_2})$. then we are required to check that

$$\Phi(\tau, f \circ \phi', f(b')) \circ {}^\tau\mathrm{Sh}(1, f) = \mathrm{Sh}(1, f) \circ \Phi(\tau, \phi', b')$$

as maps

$$\begin{array}{rcl} {}^\tau\mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1 U_1 b_1^{-1}} & \longrightarrow & \mathrm{Sh}({}^{(f\circ(\phi'\phi_1))\phi}G_2, Y(C_2)_{{}^{(f\circ(\phi'\phi_1))\phi}G_2})_{f(b'b_1)g^{-1}U_2 g f(b'b_1)^{-1}} \\ & = & \mathrm{Sh}(G_2, \psi_2, C_2)_{U_2, ((f\circ(\phi'\phi_1))\phi, f(b'b_1)g^{-1})}. \end{array}$$

This is true.

In the setting of part III we define

$$\alpha_t : \mathrm{Sh}_{E,\mathfrak{a}^+}(G, \psi, C, U) \xrightarrow{\sim} \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \inf_{D/E,t}(\psi), C, U)$$

to be the disjoint union of the maps

$$\begin{array}{ccc} \mathrm{Sh}_{E,\mathfrak{a}^+}(G, \psi, C, U)_{(\phi,b)} & \xrightarrow{\sim} & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \inf_{D/E,t}(\psi), C, U)_{(\inf_{D/E,t}(\phi), \boldsymbol{\nu}_\phi(t) b)} \\ \| & & \| \\ \mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{bUb^{-1}} & = & \mathrm{Sh}({}^{\inf_{D/E,t}(\phi)}G, Y(C)_{{}^{\inf_{D/E,t}(\phi)}G})_{\boldsymbol{\nu}_\phi(t) b U b^{-1} \boldsymbol{\nu}_\phi(t)^{-1}}. \end{array}$$

(Recall that $\boldsymbol{\nu}_\phi(t) \in Z({}^\psi G)(\mathbb{A}_E)$.) This is well defined because if $\gamma \in G(E)$ and $h \in {}^\psi G(\mathbb{A}^\infty)$, then

$$\alpha_t \circ \alpha_{(\phi,b),({}^\gamma\phi, \gamma b h^{-1})} = \alpha_{(\inf_{3,D/E,t}(\phi), \boldsymbol{\nu}_\phi(t) b), ({}^\gamma \inf_{3,D/E,t}(\phi), \gamma \boldsymbol{\nu}_\phi(t) b h^{-1})} \circ \alpha_t$$

as maps

$$\begin{array}{rcl} \mathrm{Sh}_{E,\mathfrak{a}^+}(G, \psi, C, U)_{(\phi,b)} & \longrightarrow & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \inf_{D/E,t}(\psi), C, U)_{(\inf_{D/E,t}({}^\gamma\phi), \boldsymbol{\nu}_{{}^\gamma\phi}(t)\gamma b h^{-1})} \\ & = & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \inf_{D/E,t}(\psi), C, U)_{({}^\gamma \inf_{D/E,t}(\phi), \gamma \boldsymbol{\nu}_\phi(t) b h^{-1})}. \end{array}$$

To see this note that the equality decodes to

$$\mathrm{Sh}(\mathrm{conj}_{\gamma b}(h)^{-1}, \mathrm{conj}_\gamma) = \mathrm{Sh}(\mathrm{conj}_{\gamma \boldsymbol{\nu}_\phi(t) b}(h)^{-1}, \mathrm{conj}_\gamma)$$

as maps

$$\mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{bUb^{-1}} \longrightarrow \mathrm{Sh}({}^{\gamma \inf_{D/E,t}(\phi)} G, Y(C)_{{}^{\gamma \inf_{D/E,t}(\phi)}G})_{\gamma \boldsymbol{\nu}_\phi(t) bh^{-1}Uhb^{-1}\boldsymbol{\nu}_\phi(t)^{-1}\gamma^{-1}}.$$

This is equivalent to the equality

$$\mathrm{Sh}(bh^{-1}b^{-1}, 1) = \mathrm{Sh}(\boldsymbol{\nu}_\phi(t) bh^{-1}b^{-1}\boldsymbol{\nu}_\phi(t)^{-1}, 1)$$

as maps

$$\mathrm{Sh}({}^\phi G, Y(C)_{{}^\phi G})_{bUb^{-1}} \longrightarrow \mathrm{Sh}({}^{\inf_{D/E,t}(\phi)} G, Y(C)_{{}^{\inf_{D/E,t}(\phi)}G})_{\boldsymbol{\nu}_\phi(t) bh^{-1}Uhb^{-1}\boldsymbol{\nu}_\phi(t)^{-1}},$$

which is clear as $\boldsymbol{\nu}_\phi(t)$ is central.

To see that α_t is a natural isomorphism we must check that

$$\alpha_t \circ \mathrm{Sh}_{E,\mathfrak{a}^+}(\phi, g, f) = \mathrm{Sh}_{D,\mathfrak{a}_D^+}(\inf_{D/E,t}(\phi), \boldsymbol{\nu}_\phi(t)^{-1}g, f) \circ \alpha_t.$$

On $\mathrm{Sh}_{E,\mathfrak{a}^+}(G_1, \psi_1, C_1, U_1)_{(\phi_1, b_1)}$ this equality becomes $\mathrm{Sh}(1,f) = \mathrm{Sh}(1,f)$ as maps

$$\mathrm{Sh}_{E,\mathfrak{a}^+}(G_1, \psi_1, C_1, U_1)_{(\phi_1, b_1)} \longrightarrow \mathrm{Sh}_{E,\mathfrak{a}^+}(G_2, \psi_2, C_2, U_2)_{(\inf_{D/E,t}((f\circ\phi_1)\phi), \boldsymbol{\nu}_{(f\circ\phi_1)\phi}(t) f(b_1) g^{-1})},$$

i.e. as maps from $\mathrm{Sh}({}^{\phi_1}G_1, Y(C_1)_{{}^{\phi_1}G_1})_{b_1U_1b_1^{-1}}$ to

$$\mathrm{Sh}({}^{(\inf_{D/E,t}((f\circ\phi_1)\phi)}G_2, Y(C_2)_{{}^{(\inf_{D/E,t}((f\circ\phi_1)\phi)}G_2})_{\boldsymbol{\nu}_{(f\circ\phi_1)\phi}(t) f(b_1) g^{-1}) U_2 (\boldsymbol{\nu}_{(f\circ\phi_1)\phi}(t) f(b_1) g^{-1})^{-1}}.$$

To verify that $\alpha_t \circ \Phi_{E,\mathfrak{a}^+}(\tau) = \Phi_{D,\mathfrak{a}_D^+}(\tau)\circ\alpha_t$, we must check that on ${}^\tau\mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C,U)_{(\phi_1,b_1)}$ we have

$$\Phi_{E,\mathfrak{a}^+}(\tau,\phi,b) = \Phi_{D,\mathfrak{a}_D^+}(\tau, \inf_{D/E,t}\phi, \boldsymbol{\nu}_\phi(t)b)$$

as maps

$$\begin{array}{rcl} {}^\tau\mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C,U)_{(\phi_1,b_1)} & \longrightarrow & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G,\psi,{}^\tau C, U)_{(\inf_{D/E,t}(\phi\phi_1), \boldsymbol{\nu}_{\phi\phi_1}(t) b b_1)} \\ & = & \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G, \inf_{D/E,t}(\psi), {}^\tau C, U)_{(\inf_{D/E,t}(\phi)\inf_{D/E,t}(\phi_1), \boldsymbol{\nu}_\phi(t) b \boldsymbol{\nu}_{\phi_1}(t) b_1)}, \end{array}$$

i.e. as maps

$$\begin{array}{rcl} {}^\tau\mathrm{Sh}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})_{b_1Ub_1^{-1}} & \longrightarrow & \mathrm{Sh}({}^{\inf_{D/E,t}(\phi\phi_1)}G, Y({}^\tau C)_{{}^{\inf_{D/E,t}(\phi\phi_1)}G})_{\boldsymbol{\nu}_{\phi\phi_1}(t) b b_1 U b_1^{-1} b^{-1} \boldsymbol{\nu}_{\phi\phi_1}(t)^{-1}} \\ & = & \mathrm{Sh}({}^{\phi\phi_1}G, Y({}^\tau C)_{{}^{\phi\phi_1}G})_{bb_1Ub_1^{-1}b^{-1}} \\ & = & \mathrm{Sh}({}^{\inf_{D/E,t}(\phi)\phi_1}G, Y({}^\tau C)_{{}^{\inf_{D/E,t}(\phi)\phi_1}G})_{\boldsymbol{\nu}_\phi(t) b b_1 U b_1^{-1} b^{-1} \boldsymbol{\nu}_\phi(t)^{-1}}, \end{array}$$

where $(\tau, \phi, b) \in \mathrm{Conj}_{E,\mathfrak{a}}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})$, so that

$$(\tau, \inf_{D/E,t}\phi, \boldsymbol{\nu}_\phi(t)b) \in \mathrm{Conj}_{D,\mathfrak{a}_D}({}^{\inf_{D/E,t}\phi_1}G, Y(C)_{{}^{\inf_{D/E,t}\phi_1}G}).$$

this is part (6) of theorem 3.5.

To verify that $\alpha_{t\eta_{D/E}(t')} = \alpha_{t'} \circ \alpha_t$ we must check that if $(\phi_1, b_1) \in \mathrm{Label}_{E,\mathfrak{a}}(G,\psi,C,U)$, then $\alpha_{t\eta_{D/E}(t')} = \alpha_{t'} \circ \alpha_t$ as maps

$$\begin{array}{rl} & \mathrm{Sh}_{E,\mathfrak{a}^+}(G,\psi,C,U)_{(\phi_1,b_1)} \\ \longrightarrow & \mathrm{Sh}_{D',\mathfrak{a}^+_{D'}}(G, \mathrm{inf}_{D'/E,t\eta_{D/E}(t')}\,\psi, C, U)_{(\mathrm{inf}_{D'/E,t\eta_{D/E}(t')}(\phi_1), \boldsymbol{\nu}_{\phi_1}(t\eta_{D/E}(t'))b_1)} \\ = & \mathrm{Sh}_{D',\mathfrak{a}^+_{D'}}(G, \mathrm{inf}_{D'/D,t'}\,\mathrm{inf}_{D/E,t}\,\psi, C, U)_{(\mathrm{inf}_{D'/D,t'}\,\mathrm{inf}_{D/E,t}(\phi_1), \boldsymbol{\nu}_{\mathrm{inf}_{D/E,t}(\phi_1)}(t')\boldsymbol{\nu}_{\phi_1}(t)b_1)}. \end{array}$$

Note that $\mathrm{inf}^?_{D'/E,t\eta_{D/E}(t')} = \mathrm{inf}^?_{D'/D,t'} \circ \mathrm{inf}^?_{D/E,t}$ and that $\mathrm{loc}_{\mathfrak{a}_D}\,\mathrm{inf}_{D/E,t}(\phi_1))|_{T_{2,D}(\mathbb{A}_D)} = \mathrm{loc}_{\mathfrak{a}}(\phi_1)|_{T_{2,D}(\mathbb{A}_D)} \circ \eta_{D/E}$. This is equivalent to checking that the composite of the two identity maps

$$\begin{array}{rl} & \mathrm{Sh}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})_{\mathrm{conj}_{b_1}(U)} \\ \longrightarrow & \mathrm{Sh}({}^{\mathrm{inf}_{D/E,t}\,\phi_1}G, Y(C)_{{}^{\mathrm{inf}_{D/E,t}\,\phi_1}G})_{\mathrm{conj}_{\boldsymbol{\nu}_{\phi_1}(t)b_1}(U)} \\ \longrightarrow & \mathrm{Sh}({}^{\mathrm{inf}_{D'/D,t'}\,\mathrm{inf}_{D/E,t}\,\phi_1}G, Y(C)_{{}^{\mathrm{inf}_{D'/D,t'}\,\mathrm{inf}_{D/E,t}\,\phi_1}G})_{\mathrm{conj}_{\boldsymbol{\nu}_{\mathrm{inf}_{D/E,t}\,\phi_1}(t')\boldsymbol{\nu}_{\phi_1}(t)b_1}(U)} \end{array}$$

equals the identity map

$$\begin{array}{rl} & \mathrm{Sh}({}^{\phi_1}G, Y(C)_{{}^{\phi_1}G})_{\mathrm{conj}_{b_1}(U)} \\ \longrightarrow & \mathrm{Sh}({}^{\mathrm{inf}_{D'/E,t\eta_{D/E}(t')}\,\phi_1}G, Y(C)_{{}^{\mathrm{inf}_{D'/E,t\eta_{D/E}(t')}\,\phi_1}G})_{\mathrm{conj}_{\boldsymbol{\nu}_{\phi_1}(t\eta_{D/E}(t'))b_1}(U)}, \end{array}$$

which of course it does.

If U is sufficiently small and $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$, then we define a map of complex analytic spaces

$$\begin{array}{rcl} \pi_{(\phi,b)} : {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}} \backslash (\widetilde{G}_{E,\psi}(\mathbb{A}^\infty)/U \times Y(C)_{{}^{\phi}G}) & \longrightarrow & \mathrm{Sh}(G,\psi,C,U)(\mathbb{C}) \\ (\widetilde{g}, \mu) & \longmapsto & \mathrm{Sh}(\widetilde{g}^{-1})\,{}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu), \end{array}$$

where

$${}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu) \in \mathrm{Sh}(G,\psi,C,\widetilde{g}U\widetilde{g}^{-1})_{(\phi,b)}(\mathbb{C}) = \mathrm{Sh}({}^{\phi}G, Y(C)_{{}^{\phi}G})_{b\widetilde{g}U\widetilde{g}^{-1}b^{-1}}(\mathbb{C}).$$

Equivalently

$$\begin{array}{rcl} \pi_{(\phi,b)}(\widetilde{g}, \mu) & = & {}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu) \\ & \in & \mathrm{Sh}(G,\psi,C,U)_{\widetilde{g}^{-1}(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G, Y(C)_{{}^{\phi}G})_{b\widetilde{g}U\widetilde{g}^{-1}b^{-1}}(\mathbb{C}). \end{array}$$

To see it is well defined, suppose that $\gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}}$ and $\mathrm{ad}\,\gamma$ has a lift $\widehat{\gamma} \in {}^{\phi}G(\mathbb{R})$, so that $i_{(\phi,b)}(\gamma^{-1}) = ({}^{\gamma}1, (b^{-1}\gamma^{-1}b, \gamma^{-1}\widehat{\gamma}))$. Then we must check that

$$\mathrm{Sh}({}^{\gamma}1, (b^{-1}\gamma^{-1}b, \gamma^{-1}\widehat{\gamma}), 1)({}^{\phi}G(\mathbb{Q})(b(b^{-1}\gamma b)\widetilde{g}U\widetilde{g}^{-1}(b^{-1}\gamma b)^{-1}b^{-1}, \mathrm{conj}_{\widehat{\gamma}} \circ \mu))$$

equals $({}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1}, \mu))$ in $\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})$. Here

$$\begin{array}{rcl} {}^{\phi}G(\mathbb{Q})(b(b^{-1}\gamma b)\widetilde{g}U\widetilde{g}^{-1}(b^{-1}\gamma b)^{-1}b^{-1}, \mathrm{conj}_{\widehat{\gamma}} \circ \mu) & \in & \mathrm{Sh}(G,\psi,C,b^{-1}\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1}b)_{(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G, Y(C)_{{}^{\phi}G})_{\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1}}(\mathbb{C}). \end{array}$$

The former expression equals

$$\begin{array}{rcl}({}^{\phi}G(\mathbb{Q})(\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1},\mathrm{conj}_{\widehat{\gamma}}\circ\mu)) & \in & \mathrm{Sh}(G,\psi,C,\widetilde{g}U\widetilde{g}^{-1})_{({}^{\gamma}\phi,\gamma b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\gamma\phi}G,Y(C)_{{}^{\gamma\phi}G})_{\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1}},\end{array}$$

so we must check that

$$\alpha_{({}^{\gamma}\phi,\gamma b),(\phi,b)}({}^{\phi}G(\mathbb{Q})(\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1},\mathrm{conj}_{\widehat{\gamma}}\circ\mu))=({}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1},\mu))\in\mathrm{Sh}(G,\psi,C,\widetilde{g}U\widetilde{g}^{-1})_{(\phi,b)}(\mathbb{C}),$$

i.e. that

$$\begin{array}{rcl}\mathrm{Sh}(1,\mathrm{conj}_{\gamma^{-1}})({}^{\phi}G(\mathbb{Q})(\gamma b\widetilde{g}U\widetilde{g}^{-1}b^{-1}\gamma^{-1},\mathrm{conj}_{\widehat{\gamma}}\circ\mu)) & = & ({}^{\phi}G(\mathbb{Q})(b\widetilde{g}U\widetilde{g}^{-1}b^{-1},\mu)) \\ & \in & \mathrm{Sh}(G,\psi,C,\widetilde{g}U\widetilde{g}^{-1})_{(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{b\widetilde{g}U\widetilde{g}^{-1}b^{-1}},\end{array}$$

which is clear.

We next prove that $\pi_{(\phi,b)}$ is an isomorphism. If we write

$$\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})=\coprod_{i\in I}[(\zeta_i,g_i)]\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})^1$$

then (by lemma 4.2)

$$\begin{array}{rcl}I & \stackrel{\sim}{\longrightarrow} & \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)/\sim \\ i & \longmapsto & [(\zeta_i\phi,bg_i^{-1})]\end{array}$$

and

$$\begin{array}{rl} & {}^{\phi}G^{\mathrm{ad}}(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})/U\times Y(C)_{{}^{\phi}G}) \\ = & \coprod_{i\in I}{}^{\phi}G^{\mathrm{ad}}(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{E,\psi}(\mathbb{A}^{\infty})^1/\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G}) \\ = & \coprod_{i\in I}{}^{\phi}G^{\mathrm{ad}}(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash({}^{\phi}G^{\mathrm{ad}}(E)_{\mathbb{R}}^{\mathbb{Q}}{}^{\psi}G(\mathbb{A}^{\infty})/Z(G)(\mathbb{Q})\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G}) \\ = & \coprod_{i\in I}{}^{\phi}G(\mathbb{Q})\backslash({}^{\psi}G(\mathbb{A}^{\infty})/\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G}).\end{array}$$

The map $\pi_{(\phi,b)}$ sends

$$[(g\mathrm{conj}_{g_i^{-1}}(U),\mu)]\in{}^{\phi}G(\mathbb{Q})\backslash({}^{\psi}G(\mathbb{A}^{\infty})/\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G})$$

to a point in

$$\begin{array}{rcl}\mathrm{Sh}([(\zeta_i^{-1},gg_i^{-1})])^{-1}\mathrm{Sh}(G,\psi,C,gg_i^{-1}Ug_ig^{-1})_{(\phi,b)}(\mathbb{C}) & = & \mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bgg_i^{-1})}(\mathbb{C}) \\ & = & \mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bg_i^{-1}(g_igg_i^{-1}))}(\mathbb{C}) \\ & = & \mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bg_i^{-1})}(\mathbb{C}).\end{array}$$

Thus $\pi_{(\phi,b)}$ is a disjoint union of maps

$${}^{\phi}G(\mathbb{Q})\backslash({}^{\psi}G(\mathbb{A}^{\infty})/\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G})\longrightarrow\mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bg_i^{-1})}(\mathbb{C}),$$

given by

$$\begin{array}{rcl}[(g\mathrm{conj}_{g_i}(U),\mu)] & \longmapsto & \alpha_{(\zeta_i\phi,bgg_i^{-1}),(\zeta_i\phi,bg_i^{-1})}\mathrm{Sh}([(\zeta_i^{-1},gg_i^{-1})])^{-1}{}^{\phi}G(\mathbb{Q})(bgg_i^{-1}Ug_ig^{-1}b^{-1},\mu) \\ & = & \mathrm{Sh}(\mathrm{conj}_{bg_i^{-1}}(g_ig^{-1}g_i^{-1}))({}^{\phi}G(\mathbb{Q})(bgg_i^{-1}Ug_ig^{-1}b^{-1},\mu)) \\ & = & {}^{\phi}G(\mathbb{Q})((bgb^{-1})bg_i^{-1}Ug_ib^{-1},\mu),\end{array}$$

where in the first line

$${}^{\phi}G(\mathbb{Q})(bgg_i^{-1}Ug_ig^{-1}b^{-1},\mu) \in \mathrm{Sh}(G,\psi,C,gg_i^{-1}Ug_ig^{-1})_{(\phi,b)}(\mathbb{C}) = \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bgg_i^{-1}Ug_ig^{-1}b^{-1}}(\mathbb{C})$$

and in the second line

$${}^{\phi}G(\mathbb{Q})(bgg_i^{-1}Ug_ig^{-1}b^{-1},\mu) \in \mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bgg_i^{-1})}(\mathbb{C}) = \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bgg_i^{-1}Ug_ig^{-1}b^{-1}}(\mathbb{C})$$

and in the third line

$${}^{\phi}G(\mathbb{Q})((bgb^{-1})bg_i^{-1}Ug_ib^{-1},\mu) \in \mathrm{Sh}(G,\psi,C,U)_{(\zeta_i\phi,bg_i^{-1})}(\mathbb{C}) = \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bg_i^{-1}Ug_ib^{-1}}(\mathbb{C}).$$

Thus

$$\begin{array}{rcl} \pi_{(\phi,b)} : {}^{\phi}G(\mathbb{Q})\backslash({}^{\psi}G(\mathbb{A}^\infty)/\mathrm{conj}_{g_i^{-1}}(U)\times Y(C)_{{}^{\phi}G}) & \stackrel{\sim}{\longrightarrow} & {}^{\phi}G(\mathbb{Q})\backslash({}^{\phi}G(\mathbb{A}^\infty)/\mathrm{conj}_{bg_i^{-1}}(U)\times Y(C)_{{}^{\phi}G}) \\ & \stackrel{\sim}{\longrightarrow} & \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bg_i^{-1}Ug_ib^{-1}}(\mathbb{C}), \end{array}$$

where the first map is conjugation by b. We conclude that $\pi_{(\phi,b)}$ is indeed an isomorphism.

To verify 10 note that both sides equal

$$\mathrm{Sh}(\widetilde{g}_1)\mathrm{Sh}(\widetilde{g}_2^{-1}){}^{\phi}G(\mathbb{Q})(b\widetilde{g}_2U\widetilde{g}_2^{-1}b^{-1},\mu) = \mathrm{Sh}(\widetilde{g}_1\widetilde{g}_2^{-1}){}^{\phi}G(\mathbb{Q})(b\widetilde{g}_2\widetilde{g}_1^{-1}(\widetilde{g}_1U\widetilde{g}_1^{-1})\widetilde{g}_1\widetilde{g}_2^{-1}b^{-1},\mu)$$

where

$${}^{\phi}G(\mathbb{Q})(b\widetilde{g}_2U\widetilde{g}_2^{-1}b^{-1},\mu) \in \mathrm{Sh}(G,\psi,C,\widetilde{g}_2U\widetilde{g}_2^{-1})_{(\phi,b)}(\mathbb{C}) = \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{b\widetilde{g}_2U\widetilde{g}_2^{-1}b^{-1}}(\mathbb{C}).$$

To verify 11 we may (using part 10) suppose that $\widetilde{g}=1$. In this case left hand side is represented by

$$\begin{array}{rcl} {}^{\gamma\phi}G(\mathbb{Q})(\gamma bUb^{-1}\gamma^{-1},\mathrm{conj}_\gamma(\mu)) & \in & \mathrm{Sh}(G,\psi,C,U)_{({}^{\gamma}\phi,\gamma b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\gamma\phi}G,Y(C)_{{}^{\gamma\phi}G})_{\gamma bUb^{-1}\gamma^{-1}}(\mathbb{C}), \end{array}$$

while the right hand side is represented by

$$\begin{array}{rcl} {}^{\phi}G(\mathbb{Q})(bUb^{-1},\mu) & \in & \mathrm{Sh}(G,\psi,C,U)_{(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bUb^{-1}}(\mathbb{C}). \end{array}$$

Thus we need to verify that

$$\begin{array}{rcl} \mathrm{Sh}(1,\mathrm{conj}_\gamma)({}^{\phi}G(\mathbb{Q})(bUb^{-1},\mu)) & = & {}^{\gamma\phi}G(\mathbb{Q})(\gamma bUb^{-1}\gamma^{-1},\mathrm{conj}_\gamma(\mu)) \\ & \in & \mathrm{Sh}({}^{\gamma\phi}G,Y(C)_{{}^{\gamma\phi}G})_{\gamma bh^{-1}Uhb^{-1}\gamma^{-1}}(\mathbb{C}), \end{array}$$

which is clear.

To verify property 12 we may (using part 10) assume that $\widetilde{g}_2=1$ and that $\widetilde{g}_1=[(\zeta,g)]$. In this case the left hand side is represented by

$$\begin{array}{rcl} {}^{\zeta\phi}G(\mathbb{Q})((bg^{-1})gUg^{-1}(bg^{-1})^{-1},\mu) & \in & \mathrm{Sh}(G,\psi,C,U)_{\widetilde{g}_1^{-1}\widetilde{g}_1(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bUb^{-1}}. \end{array}$$

This equals

$$\begin{array}{rcl} {}^{\phi}G(\mathbb{Q})(bUb^{-1},\mu) & \in & \mathrm{Sh}(G,\psi,C,U)_{(\phi,b)}(\mathbb{C}) \\ & = & \mathrm{Sh}({}^{\phi}G,Y(C)_{{}^{\phi}G})_{bUb^{-1}}, \end{array}$$

which represents the right hand side.

To verify the first half of property 13 note that the left hand side is represented by the image under $\mathrm{Sh}(\phi, g, f)$ of

$$\begin{aligned} {}^{\zeta^{-1}\phi_1}G_1(\mathbb{Q})(b_1kU_1k^{-1}b_1^{-1}, \mu) &\in \mathrm{Sh}(G_1, \psi_1, C_1, U_1)_{(\zeta^{-1}\phi_1, b_1k)}(\mathbb{C}) \\ &= \mathrm{Sh}({}^{\zeta^{-1}\phi_1}G_1, Y(C_1)_{{}^{\zeta^{-1}\phi_1}G_1})_{b_1kU_1k^{-1}b_1^{-1}}, \end{aligned}$$

i.e. by

$$\begin{aligned} f({}^{f(\zeta^{-1}\phi_1)\phi}G_2(\mathbb{Q})(f(b_1k)g^{-1}U_2gf(b_1k)^{-1}, f(\mu))) &\in \mathrm{Sh}(G_2, \psi_2, C_2, U_2)_{(f(\zeta^{-1}\phi_1)\phi, f(b_1k)g^{-1})}(\mathbb{C}) \\ &= \mathrm{Sh}({}^{f(\zeta^{-1}\phi_1)\phi}G_2, Y(C_2)_{{}^{f(\zeta^{-1}\phi_1)\phi}G_2})_{f(b_1k)g^{-1}U_2gf(b_1k)^{-1}} \end{aligned}$$

which also represents the right hand side. In the case that $[(\zeta, k)] \in \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)_f$, this also equals

$$\begin{aligned} & f({}^{f(\phi_1)\phi}G_2(\mathbb{Q})(f(b_1)g^{-1}(gf(k)g^{-1})U_2(gf(k)g^{-1})^{-1}gf(b_1)^{-1}, f(\mu))) \\ \in\ & \mathrm{Sh}(G_2, \psi_2, C_2, U_2)_{\widetilde{\theta}_{(\phi,g,f)}(\widehat{k})^{-1}(f(\phi_1)\phi, f(b_1)g^{-1})}(\mathbb{C}) \\ =\ & \mathrm{Sh}({}^{f(\phi_1)\phi}G_2, Y(C_2)_{{}^{f(\phi_1)\phi}G_2})_{f(b_1k)g^{-1}U_2gf(b_1k)^{-1}}, \end{aligned}$$

which is just $\pi_{(f(\phi_1)\phi, f(b_1)g^{-1})}([(\widetilde{\theta}_{(\phi,g,f)}(\widehat{k}), f\circ\mu)])$, as desired.

To verify property (14) it suffices (by part 10) to treat the case $\widetilde{g} = 1$. Then $\pi_{(\phi,b)}(1,\mu)$ is represented by $T(\mathbb{Q})(U,\mu) \in \mathrm{Sh}(T,\{\mu\})_U = \mathrm{Sh}(T,\psi,\{\mu\},U)_{(\phi,b)}$ and $\Phi(\tau)\circ\tau\circ\pi_{(\phi,b)}(1,\mu)$ is represented by

$$(T(\mathbb{Q})(U, {}^\tau\mu)) \in \mathrm{Sh}(T,\psi,\{\mu\},U)_{(\phi_\tau\phi, b_\tau b)} = \mathrm{Sh}(T, \{{}^\tau\mu\})_U.$$

This equals $\pi_{(\phi_\tau\phi, b_\tau b)}(1, {}^\tau\mu)$.

To verify property 15, it suffices (by part 10) to show that

$$\alpha_t \circ \pi_{E,\mathfrak{a}^+;\rho,(\phi,b)}(1,\mu) = \pi_{D,\mathfrak{a}_D^+;\rho,\mathrm{inf}_{D/E,t}(\phi,b)}(1,\mu).$$

However both sides are represented by

$$\begin{aligned} {}^{\mathrm{inf}_{D/E,t}(\phi)}G(\mathbb{Q})(bUb^{-1},\mu) &\in \mathrm{Sh}_{D,\mathfrak{a}_D^+}(G,\psi,C,U)_{(\mathrm{inf}_{D/E,t}(\phi), \boldsymbol{\nu}_\phi(t)b)}(\mathbb{C}) \\ &= \mathrm{Sh}({}^{\mathrm{inf}_{D/E,t}(\phi)}G, Y(C)_{{}^{\mathrm{inf}_{D/E,t}(\phi)}G})_{\boldsymbol{\nu}_\phi(t)bUb^{-1}\boldsymbol{\nu}_\phi(t)^{-1}}. \end{aligned}$$

4.5. **The general case.** First suppose that L is a number field embeddable into E.

If $\rho : L \hookrightarrow \mathbb{C}$ then for any $\tau \in \mathrm{Aut}\,(\mathbb{C}/\rho L)$ we have

$$\Phi(\tau) : {}^\tau\mathrm{Sh}(G,\psi,{}^\rho C,U) \xrightarrow{\sim} \mathrm{Sh}(G,\psi,{}^\rho C,U),$$

and these maps provide descent data, i.e. $\Phi(\tau_1\tau_2) = \Phi(\tau_1)\circ{}^{\tau_1}\Phi(\tau_2)$ for all $\tau_1,\tau_2 \in \mathrm{Aut}\,(\mathbb{C}/\rho L)$. Note that the automorphism group of $\mathrm{Sh}(G,\psi,{}^\rho C,U)(\mathbb{C})$ is finite and that, by corollary 4.4, the set of $\mathbb{C}$ points of $\mathrm{Sh}(G,\psi,{}^\rho C,U)$ with finite orbit under $\{\Phi(\tau)\circ\tau :\ \tau\in\mathrm{Aut}\,(\mathbb{C}/L)\}$ is Zariski dense. It follows from corollary 1.2 of [Mi2] there is a unique model $\mathrm{Sh}_{L,\rho}(G,\psi,C,U)/L$ of $\mathrm{Sh}(G,\psi,C,U)$ over L (with respect to ρ). (Note that if $z\in\mathbb{C}$ is fixed by a finite index subgroup of $\mathrm{Aut}\,(\mathbb{C})$, then z is algebraic.)

The maps $\mathrm{Sh}(\phi,g,f)$ and α_t commute with $\Phi(\tau)$ for $\tau\in\mathrm{Aut}\,(\mathbb{C}/\rho L)$ and so descend to maps $\mathrm{Sh}_{L,\rho}(\phi,g,f)$ and $\alpha_{L,\rho,t}$. These satisfy properties I, III, 1, 2, 3, 4, 5, 6 and 9.

If $\rho' : L' \hookrightarrow \mathbb{C}$ and $\tau : L \to L'$, then we can find $\widetilde{\tau} \in \mathrm{Aut}\,(\mathbb{C})$ with with $\rho'\tau = \widetilde{\tau}\rho$. The element $\widetilde{\tau}$ is not unique, but the coset $\mathrm{Aut}\,(\mathbb{C}/\rho' L)\widetilde{\tau}$ is. Then

$$\Phi(\widetilde{\tau}) : {}^{\tau}\mathrm{Sh}(G, \psi, {}^{\rho}C, U) \xrightarrow{\sim} \mathrm{Sh}(G, \psi, {}^{\rho'\tau}C, U)$$

descends to

$$\Phi(\widetilde{\tau})_{\rho,\rho'} : {}^{\tau}\mathrm{Sh}_{L,\rho}(G, \psi, C, U) \stackrel{\sim}{\longrightarrow} \mathrm{Sh}_{L',\rho'}(G, \psi, {}^{\tau}C, U)$$

over L'. We have the following observations

- $\Phi(\widetilde{\tau})_{\rho,\rho'} : \tau \circ \mathrm{Sh}_{L,\rho} \xrightarrow{\sim} \mathrm{Sh}_{L',\rho'} \circ \tau$ is a natural isomorphism. It commutes with the α_t.
- $\Phi(\widetilde{\tau}')_{\rho',\rho''} \circ {}^{\tau'}\Phi(\widetilde{\tau})_{\rho,\rho'} = \Phi(\widetilde{\tau}'\widetilde{\tau})_{\rho,\rho''}$.
- If $L' = L$ and $\widetilde{\tau} \in \mathrm{Aut}\,(\mathbb{C}/\rho L)$ then $\Phi(\widetilde{\tau})_{\rho,\rho}$ is the identity. (This follows from the definition of $\mathrm{Sh}(G, \psi, {}^{\rho}C, U)_{\rho}$.)
- $\Phi(\widetilde{\tau})_{\rho,\rho'}$ depends only on the coset $\mathrm{Aut}\,(\mathbb{C}/\rho' L)\widetilde{\tau}$, and so we unambiguously may denote it $\Phi(\tau)_{\rho,\rho'}$.

Using $\Phi(1)_{\rho,\rho'}$ we can identify $\mathrm{Sh}_{L,\rho}$ with $\mathrm{Sh}_{L,\rho'}$. These identifications are compatible in triples. By Sh_L we shall mean any of the $\mathrm{Sh}_{L,\rho}$ identified in this way. If $\tau : L_1 \to L_2$ and $\rho_i, \rho_i' : L_i \hookrightarrow \mathbb{C}$ then

$$\Phi(1)_{\rho_2,\rho_2'} \circ \Phi(\tau)_{\rho_1,\rho_2} = \Phi(\tau)_{\rho_1',\rho_2'} \circ \Phi(1)_{\rho_1,\rho_1'}.$$

Thus we get a well defined natural isomorphism

$$\Phi(\tau) : \mathrm{Sh}_L \stackrel{\sim}{\longrightarrow} \mathrm{Sh}_{L'}.$$

It is easy to check that these satisfy parts 7 and 8.

Finally we treat the case of a general field L of characteristic 0. Any subfield L_1 of L embeddable into E will be contained in $E \cap L$. Thus, if $\iota_L : E \cap L \hookrightarrow L$ is the natural inclusion, then $\mathrm{RSD}(\iota_L) : \mathrm{RSD}(E, \mathfrak{a}; E \cap L) \to \mathrm{RSD}(E, \mathfrak{a}; L)$ is an isomorphism of categories (i.e. a functor which is bijective on objects and morphisms). We define $\mathrm{Sh}_{E,\mathfrak{a}^+;L} = \iota_L \circ \mathrm{Sh}_{E,\mathfrak{a}^+;E\cap L} \circ \mathrm{RSD}(\iota_L)^{-1}$. If $\tau : L \to L'$ then $\tau(E \cap L) \subset E \cap L'$ and we define $\Phi_{E,\mathfrak{a}^+}(\tau)$ by

$$\Phi_{E,\mathfrak{a}^+}(\tau) \circ \iota_L = \iota_{L'} \Phi_{E,\mathfrak{a}^+}(\tau|_{E\cap L}).$$

We also define α_t by

$$\alpha_t \circ \iota_L = \iota_{L'} \circ \alpha_t \circ \iota,$$

where $\iota : E \cap L \hookrightarrow D \cap L$ is the natural embedding. It is straight forward that all the properties of the theorem are satisfied and that this last definition coincides with the ones already made for $\mathbb{C}$ and for number fields embeddable into E.

This completes the proof of theorem 4.3.

4.6. **Local systems.** If $U \subset Z(G)(\mathbb{A}^\infty)$ is an open compact subgroup we will write $Z(G)^1_U$ for the Zariski closure of $Z(G)(\mathbb{Q}) \cap U$. Moreover write

$$Z(G)^1 = \bigcap_U Z(G)^1_U.$$

It is a torus, $(Z(G)^1_U)^0 = Z(G)^1$ for any U, and there exists U with $Z(G)^1 = Z(G)^1_U$. If V is a sufficiently small open compact subgroup of $G(\mathbb{A}^\infty)$ we will call it *more sufficiently small* if $Z(G)^1 = Z(G)^1_{V \cap Z(G)(\mathbb{A}^\infty)}$. Moreover if $\overline{Z(G)(\mathbb{Q})}$ denotes the closure of $Z(G)(\mathbb{Q})$ in $Z(G)(\mathbb{A}^\infty)$ we have

$$\overline{Z(G)(\mathbb{Q})} = \overline{Z(G)(\mathbb{Q}) \cap U} Z(G)(\mathbb{Q}) = \overline{Z(G)(\mathbb{Q})^1} Z(G)(\mathbb{Q})$$

(because $Z(G)(\mathbb{A}^\infty)/(U \cap Z(G)(\mathbb{A}^\infty))$ is discrete).

Suppose that $(G, \psi, C, U) \in RSD(E, \mathfrak{a}; L)$ is a rational Shimura datum with U more sufficiently small. Suppose moreover that $\boldsymbol{W}/\mathbb{A}^\infty$ is a finite free module and that $\boldsymbol{r} : {}^\psi G/Z(G)^1 \to GL(\boldsymbol{W})$ over $\mathbb{A}^\infty$ is a representation. As V runs over normal open subgroups of U we have continuous homomorphisms

$$\begin{array}{rcl} \lim_{\leftarrow V} \mathrm{Gal}\,(\mathrm{Sh}(G, \psi, C, V)/\mathrm{Sh}(G, \psi, C, U)) & \cong & U/(\overline{Z(G)(\mathbb{Q})} \cap U) \\ & \longrightarrow & ({}^\psi G/Z(G)^1)(\mathbb{A}^\infty) \\ & \xrightarrow{\boldsymbol{r}} & GL(\boldsymbol{W})(\mathbb{A}^\infty). \end{array}$$

Thus we obtain a lisse etale $\mathbb{A}^\infty$-sheaf $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}/\mathrm{Sh}(G, \psi, C, U)$. The map $(\boldsymbol{W}, \boldsymbol{r}) \mapsto \boldsymbol{\mathcal{W}}_{\boldsymbol{r}}$ extends to an exact $\mathbb{A}^\infty$-linear tensor functor. In particular if $g \in {}^\psi G(\mathbb{A}^\infty)$, then $r(g)^{-1} : (\boldsymbol{W}, \boldsymbol{r} \circ \mathrm{conj}_g) \xrightarrow{\sim} (\boldsymbol{W}, \boldsymbol{r})$ induces an isomorphism

$$g^{-1} : \boldsymbol{\mathcal{W}}_{\boldsymbol{r} \circ \mathrm{conj}_g} \xrightarrow{\sim} \boldsymbol{\mathcal{W}}_{\boldsymbol{r}}.$$

If $(\phi, g, f) : (G_1, \psi_1, C_1, U_1) \to (G_2, \psi_2, C_2, U_2)$ is a morphism in $\mathrm{RSD}(E, \mathfrak{a}; L)$ and if $(\boldsymbol{W}_2, \boldsymbol{r}_2)$ is a representation of ${}^{\psi_2} G_2$ on a finite free $\mathbb{A}^\infty$-module, then $\boldsymbol{r}_2 \circ \mathrm{conj}_g \circ f$ is a representation of ${}^{\psi_1} G$ on $\boldsymbol{W}_2$. Moreover

$$\mathrm{Sh}(\phi, g, f)^* \boldsymbol{\mathcal{W}}_{\boldsymbol{r}_2} = \boldsymbol{\mathcal{W}}_{\boldsymbol{r}_2 \circ \mathrm{conj}_g \circ f}.$$

Suppose that $(\zeta, g) \in \widetilde{G}_{E,\psi}(\mathbb{A}^\infty)$, $(G, \psi, C, U) \in \mathrm{RSD}(E, \mathfrak{a}; L)$ and $(\boldsymbol{W}, \boldsymbol{r})$ is a representation of ${}^\psi G$ over $\mathbb{A}^\infty$. Set

$${}^\zeta \boldsymbol{W} = \{ w \in \boldsymbol{W} \otimes_{\mathbb{Q}} E : {}^{1 \otimes \sigma} w = \zeta(\sigma)^{-1} w \ \forall \sigma \in \mathrm{Gal}\,(E/\mathbb{Q}) \} = \boldsymbol{r}(g)^{-1} W,$$

a finite free $\mathbb{A}^\infty$-module. Then $\boldsymbol{r}$ gives a map

$${}^\psi G \longrightarrow GL({}^\zeta \boldsymbol{W}/\mathbb{A}^\infty)$$

over $\mathbb{A}^\infty$ which we will denote ${}^\zeta \boldsymbol{r}$. Then we get a ${}^\psi G$-equivariant map

$$\boldsymbol{r}(g)^{-1} : (\boldsymbol{W}, \boldsymbol{r} \circ \mathrm{conj}_g) \xrightarrow{\sim} (\boldsymbol{r}(g)^{-1} \boldsymbol{W}, \boldsymbol{r}) = ({}^\zeta \boldsymbol{W}, \boldsymbol{r}),$$

and hence a map

$$g^{-1} : \mathrm{Sh}(\zeta, g)^* \boldsymbol{\mathcal{W}}_{\boldsymbol{r}} \longrightarrow \boldsymbol{\mathcal{W}}_{{}^\zeta \boldsymbol{r}}.$$

This in turn gives

$$g^{-1}: \mathrm{Sh}(\zeta, g)^*(\boldsymbol{\mathcal{W}}_{\boldsymbol{r}} \otimes_{\mathbb{Q}} E) \longrightarrow (\boldsymbol{\mathcal{W}}_{\boldsymbol{r}} \otimes_{\mathbb{Q}} E).$$

For the rest of this section suppose that $L = \mathbb{C}$. We will write $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}^{an}$ for the locally constant sheaf of $\mathbb{A}^\infty$-modules on $\mathrm{Sh}(G, \psi, C, U)(\mathbb{C})$ corresponding to $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}$.

We define

$$\mathrm{Label}_{E,\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r}) = \mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})$$

to be the set of tuples (ϕ, b, W, r, α), where $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, where $W/\mathbb{Q}$ is a vector space, $r : {}^\phi G \to GL(W/\mathbb{Q})$ is a representation defined over $\mathbb{Q}$ and

$$\alpha : W \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} \boldsymbol{W}$$

over $\mathbb{A}^\infty$ such that

$$\boldsymbol{r} \circ \mathrm{conj}_b^{-1} = \mathrm{conj}_\alpha \circ r.$$

This set may be empty. If it is non-empty, we will call $\boldsymbol{r}$ *rationalizable.*

If $(\phi, b) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C)$, if r is a representation of ${}^\phi G$ on a $\mathbb{Q}$-vector space W, and if $\zeta \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E))$, then we set

$${}^\zeta W = \{w \in W \otimes_{\mathbb{Q}} E : \ {}^{1\otimes\sigma} w = \zeta(\sigma)^{-1} w \ \forall \sigma \in \mathrm{Gal}\,(E/\mathbb{Q})\}.$$

It is preserved by the action of ${}^\phi G$ (via r). If $\gamma \in {}^\phi G(E)_{\mathbb{R}}^{\mathbb{Q}}$ and $i_{(\phi,b)}(\gamma) = (\zeta, g)$, then $r(\gamma) W = {}^{\zeta^{-1}} W$.

We will call two elements (ϕ, b, W, r, α) and $(\phi', b', W', r', \alpha') \in \mathrm{Label}\,(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})$ *equivalent* if there exists $(\delta, (\eta, h)) \in G(E) \times \widetilde{G}_\psi(\mathbb{A}^\infty)$ and an isomorphism $\beta : {}^{r\circ\eta} W \xrightarrow{\sim} W'$ of $\mathbb{Q}$-vector spaces, such that

- $(\phi', b') = {}^{(\delta,(\eta,h))}(\phi, b) = ({}^\delta \phi \eta, \delta b h^\infty)$,
- and $\alpha \circ r(b h^\infty b^{-1}) = \alpha' \circ \beta$.

This implies that

- $\beta \circ r(\gamma) = r'(\delta \gamma \delta^{-1}) \circ \beta$.

If $((\delta', (\eta', h')), \beta')$ also exhibits the equivalence of (ϕ, b, W, r, α) and $(\phi', b', W', r', \alpha')$, then $\delta^{-1}\delta' \in {}^\phi G(E)_{\mathbb{R}}^{\mathbb{Q}}$ and $(\eta', h') = (\eta, h) i_{(\phi,b)}(\delta^{-1}\delta')$ and $\beta' = \beta \circ r(\delta^{-1}\delta')$.

Write $Z(\boldsymbol{r})$ for the centralizer in $GL(\boldsymbol{W})$ of $\boldsymbol{r}$. The set $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})$ has an action of $Z(\boldsymbol{r})(\mathbb{A}^\infty)$ via

$$z(\phi, b, W, r, \alpha) = (\phi, b, W, r, z \circ \alpha).$$

This action preserves the equivalence relation $\sim$ and so descends to an action on $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})/\sim$.

Lemma 4.5. *(1) The action of $Z(\boldsymbol{r})$ on $\mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})/\sim$ is transitive.*

(2) The stabilizer in $Z(\boldsymbol{r})$ of $[(\phi, b, W, r, \alpha)] \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})/\sim$ is $\mathrm{conj}_\alpha(Z(r))$, where $Z(r)$ denotes the centralizer in $GL(W)$ of r.

Proof: For the first part note that, as the action of $G(E)\times\widetilde{G}_\psi(\mathbb{A}^\infty)$ on $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$ is transitive, it suffices to show that if (ϕ,b,W,r,α) and (ϕ,b,W',r',α') are in the set $\mathrm{Label}_{\mathfrak{a}}(G,\psi,C,\boldsymbol{W},\boldsymbol{r})$ then there is an isomorphism $\beta: W \stackrel{\sim}{\to} W'$ such that $\mathrm{conj}_\beta \circ r = r'$. To see such a β exists chose an isomorphism of $\mathbb{Q}$-vector spaces $f: W' \stackrel{\sim}{\to} W$ and look at the $\mathbb{Q}$-vector space $H = \mathrm{Hom}_{{}^\phi G}(W,W')$ together with the polynomial function $h \mapsto \det(f\circ h)$. As the polynomial function does not vanish identically on $H\otimes_{\mathbb{Q}}\mathbb{A}^\infty$, it is not the zero polynomial and hence does not vanish identically on $\mathbb{Q}$, as desired.

For the second part, if $(\delta,(\eta,h)),\beta)$ exhibits an equivalence between (ϕ,b,W,r,α) and $(\phi,b,W,r,z\alpha)$, then $\delta\in {}^\phi G(E)^{\mathbb{Q}}_{\mathbb{R}}$ and $(\eta,h)=i_{(\phi,b)}(\delta)$ and $\beta: r(\delta)^{-1}W \stackrel{\sim}{\to} W$ (i.e. $\beta r(\delta)^{-1}\in Z(r)$) and $\alpha\circ r(\delta) = z\circ\alpha\circ\beta$, i.e. $z=\mathrm{conj}_\alpha(r(\delta)\beta^{-1})$, as desired. □

If $(\phi,b,W,r,\alpha)\sim(\phi',b',W',r',\alpha')$ and this equivalence is exhibited by $((\delta,(\eta,h)),\beta)$ the $\mathrm{conj}_\beta: Z(r)\stackrel{\sim}{\to} Z(r')$. This isomprphism does not depend on the choice of $((\delta,(\eta,h)),\beta)$. Thus if $\lambda\in$ we may write without ambiguity $Z(\lambda)$ for $Z(r)$ for any $(\phi,b,W,r,\alpha)\in\lambda$.

Construction Summary. *Suppose that $\boldsymbol{r}$ is rationalizable and $\lambda\in\mathrm{Label}_{\mathfrak{a}}(G,\psi,C,\boldsymbol{W},\boldsymbol{r})/\sim$. Then there is a canonically defined locally constant sheaf of $\mathbb{Q}$-modules $\mathcal{W}_{\boldsymbol{r},\lambda}$ on* $\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})$ *together with*

- *an isomorphism $\widetilde{\alpha}_\lambda: \mathcal{W}_{\boldsymbol{r},\lambda}\otimes_{\mathbb{Q}}\mathbb{A}^\infty \stackrel{\sim}{\to} \boldsymbol{\mathcal{W}}_{\boldsymbol{r}}$;*
- *a decreasing (exhaustive and separating) filtration Fil^i on $\mathcal{W}_{\boldsymbol{r},\lambda}\otimes_{\mathbb{Q}}\mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}$ satisfying Griffiths transversality (i.e. $(1\otimes d)\mathrm{Fil}^i\mathcal{W}_{\boldsymbol{r}}\otimes_{\mathbb{Q}}\mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}\subset \mathrm{Fil}^{i-1}\mathcal{W}_{\boldsymbol{r}}\otimes_{\mathbb{Q}}\Omega^1_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}$), which makes $(\mathcal{W}_{\boldsymbol{r}},\{\mathrm{Fil}^i\})$ a variation of Hodge structures with $\mathcal{W}_{\boldsymbol{r}}\otimes_{\mathbb{Q}}\mathbb{R}$ is polarizable.*

Given $\lambda\in\mathrm{Label}_{E,\mathfrak{a}}(G,\psi,C,\boldsymbol{W},\boldsymbol{r})/\sim$ and $z\in Z(\boldsymbol{r})$ there is a unique isomorphism

$$(\mathcal{W}_{\boldsymbol{r},\lambda},\{\mathrm{Fil}\})\cong(\mathcal{W}_{\boldsymbol{r},z\lambda},\{\mathrm{Fil}\})$$

such that

$$\begin{array}{ccc} \mathcal{W}_{\boldsymbol{r},\lambda} & \cong & \mathcal{W}_{\boldsymbol{r},z\lambda} \\ \widetilde{\alpha}_\lambda\downarrow & & \downarrow\widetilde{\alpha}_{z\lambda} \\ \boldsymbol{\mathcal{W}}_{\boldsymbol{r}} & \stackrel{\longrightarrow}{z} & \boldsymbol{\mathcal{W}}_{\boldsymbol{r}} \end{array}$$

commutes. (However the choice of z taking λ to $\lambda'=z\lambda$ is not unique.) Thus $\mathcal{W}_{\boldsymbol{r},\lambda}$ is independent of λ, but only up to an isomorphism that is unique only up to composition with an element of $Z(\lambda)$.

To carry out this construction we must first give a more direct description of $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}^{an}$.

Lemma 4.6. *${}^\phi G(E)^{\mathbb{Q}}_{\mathbb{R}}\backslash(\widetilde{G}_\psi(\mathbb{A}^\infty)/\overline{Z(G)^1(\mathbb{Q})}\times Y(C)_{{}^\phi G})$ maps homeomorphically to*

$$\lim_{\leftarrow V}{}^\phi G(E)^{\mathbb{Q}}_{\mathbb{R}}\backslash(\widetilde{G}_\psi(\mathbb{A}^\infty)/V\times Y(C)_{{}^\phi G}).$$

Proof: Note that

$$ {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/V \times Y(C)_{{}^{\phi}G}) = {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/\overline{Z(G)^{1}(\mathbb{Q})}V \times Y(C)_{{}^{\phi}G}), $$

so the map is well defined. It is continuous, and open by the definitions of the quotient and inverse image topologies. It is clearly surjective, so we need only check it is injective. So suppose that $[(g_1,\mu_1)]$ and $[(g_2,\mu_2)] \in {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/\overline{Z(G)^{1}(\mathbb{Q})} \times Y(C)_{{}^{\phi}G})$ have the same image in ${}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/V \times Y(C)_{{}^{\phi}G})$. We must show that $[(g_1,\mu_1)] = [(g_2,\mu_2)]$. As they become equal in ${}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/U \times Y(C)_{{}^{\phi}G})$ then we can find $\gamma \in {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}$ and $u \in U$ with $(\gamma g_1 u, \gamma\mu_1) = (g_2,\mu_2)$. If $\gamma' \in {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}$ and $u' \in U$ also have this property then $\gamma^{-1}\gamma' \in {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}} \cap U$ fixes mu_1. As U is sufficiently small we deduce that $\gamma^{-1}\gamma' \in Z(G)(E) \cap U = Z(G)(\mathbb{Q}) \cap U$. Thus for any open subgroup $V \subset U$ we have $(\gamma\delta_V g_1 v_V, \gamma\mu_1) = (g_2,\mu_2)$ for some $\delta_V \in Z(G)(\mathbb{Q})$ and $v_V \in V$ so that

$$ g_1^{-1}\gamma^{-1}g_2 \in \bigcap_V Z(G)(\mathbb{Q})V = \overline{Z(G)(\mathbb{Q})} = \overline{Z(G)^{1}(\mathbb{Q})}Z(G)(\mathbb{Q}). $$

The result follows. □

As a consequence we see that, if $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$, then we have

$$ \boldsymbol{\mathcal{W}}_{\boldsymbol{r}}^{an}(\Omega) = \{f : \widetilde{\Omega} \to \boldsymbol{W} :\ f \text{ continuous, and } f(xu^{-1}) = \boldsymbol{r}(u)f(x)\ \forall u \in U,\ x \in \widetilde{\Omega}\}, $$

where $\widetilde{\Omega}$ denotes the preimage of $\pi_{(\phi,b)}^{-1}\Omega \subset {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/U \times Y(C)_{{}^{\phi}G})$ in the space ${}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/\overline{Z(G)^{1}(\mathbb{Q})} \times Y(C)_{{}^{\phi}G})$. If $(\delta,(\eta,h)) \in G(E) \times \widetilde{G}_{\psi}(\mathbb{A}^{\infty})$ then $\pi^{-1}_{{}^{(\delta,(\eta,h))}(\phi,b)} = ((\eta,h),\mathrm{conj}_{\delta})\pi^{-1}_{(\phi,b)}$, and so the descriptions of $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}^{an}(\Omega)$ for (ϕ,b) and ${}^{(\delta,(\eta,h))}(\phi,b)$ are related by the identification of

$$ \{f : \widetilde{\Omega} \to \boldsymbol{W} :\ f \text{ continuous, and } f(xu^{-1}) = \boldsymbol{r}(u)f(x)\ \forall u \in U,\ x \in \widetilde{\Omega}\} $$

with

$$ \{f' : ((\eta,h),\mathrm{conj}_{\delta})\widetilde{\Omega} \to \boldsymbol{W} :\ f' \text{ continuous, and } f'(yu^{-1}) = \boldsymbol{r}(u)f'(y)\ \forall u \in U,\ y \in ((\eta,h),\mathrm{conj}_{\delta})\widetilde{\Omega}\} $$

via

$$ f'(y) = f(((\eta,h),\mathrm{conj}_{\delta})^{-1}y). $$

Lemma 4.7. *If $(\phi,b) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C)$, then we have a description of $\boldsymbol{\mathcal{W}}_{\boldsymbol{r}}$ given by*

$$ \boldsymbol{\mathcal{W}}_{\boldsymbol{r}}^{an}(\Omega) = \{F : \widetilde{\Omega} \to \boldsymbol{W} \otimes_{\mathbb{Q}} E :\ F \text{ locally constant; and } F((\zeta,g),\mu) \in {}^{\zeta^{-1}}\boldsymbol{W}; \text{ and } F((i_{(\phi,b)}(\gamma),\mathrm{conj}_{\gamma})x) = \boldsymbol{r}(b^{-1}\gamma b)F(x)\ \forall \gamma \in {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}},\ x \in \widetilde{\Omega}\}, $$

with the obvious restriction maps, where $\widetilde{\Omega}$ denotes the preimage of $\pi_{(\phi,b)}^{-1}\Omega \subset {}^{\phi}G(E)_{\mathbb{R}}^{\mathbb{Q}}\backslash(\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/U \times Y(C)_{{}^{\phi}G})$ in the space $\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/\overline{Z(G)^{1}(\mathbb{Q})}U \times Y(C)_{{}^{\phi}G}$.

If instead we use ${}^{(\delta,(\eta,h))}(\phi,b)$ *we have*

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{F' : ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega} \to \boldsymbol{W} \otimes_{\mathbb{Q}} E :\ F' \text{ locally constant; and}\\ F'((\zeta,g),\mu) \in \boldsymbol{r}(g)\boldsymbol{W} = {}^{\zeta^{-1}}\boldsymbol{W};\\ \text{and } F'((i_{(\phi,b)}(\gamma),\mathrm{conj}_\gamma)x) = \boldsymbol{r}(b^{-1}\gamma b)F'(x)\ \forall\gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\}.\end{aligned}$$

The canonical identification of these two sets is via:

$$F'(y) = \boldsymbol{r}(h^\infty)F(((\eta,h),\mathrm{conj}_\delta)^{-1}y).$$

Proof: We first reinterpret our first description of $\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}$ as

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{f : \widetilde{\Omega}' \to \boldsymbol{W} :\ f \text{ continuous; and } f((i_{(\phi,b)}(\gamma),\mathrm{conj}_\gamma)xu^{-1}) = \boldsymbol{r}(u)f(x)\\ \forall u \in \overline{Z(G)^1(\mathbb{Q})}U,\ \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega};\},\end{aligned}$$

where now $\widetilde{\Omega}'$ is the preimage of $\pi^{-1}_{(\phi,b)}\Omega \subset {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}}\backslash(\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{\phi G})$ in $\widetilde{G}_\psi(\mathbb{A}^\infty) \times Y(C)_{\phi G}$. Using instead ${}^{(\delta,(\eta,h))}(\phi,b)$ we have

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{f' : ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega}' \to \boldsymbol{W} :\ f' \text{ continuous; and}\\ f'((\underline{(\eta,h)i_{(\phi,b)}}(\delta^{-1}\gamma\delta)(\eta,h)^{-1},\mathrm{conj}_\gamma)yu^{-1}) = \boldsymbol{r}(u)f'(y)\\ \forall u \in \overline{Z(G)^1(\mathbb{Q})}U,\ \gamma \in {}^{\delta\phi\eta}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ y \in ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega}\},\end{aligned}$$

i.e.

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{f' : ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega}' \to \boldsymbol{W} :\ f' \text{ continuous; and}\\ f'(((\eta,h),\mathrm{conj}_\delta)\underline{(i_{(\phi,b)}(\gamma),\mathrm{conj}_\gamma)}((\eta,h)\mathrm{conj}_\delta)^{-1}yu^{-1}) = \boldsymbol{r}(u)f'(y)\\ \forall u \in \overline{Z(G)^1(\mathbb{Q})}U,\ \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ y \in ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega}\}.\end{aligned}$$

The canonical identification of these two descriptions is via

$$f'(y) = f(((\eta,h),\mathrm{conj}_\delta)^{-1}y).$$

Writing

$$F_f((\zeta,g),\mu) = \boldsymbol{r}(g^\infty)f((\zeta,g),\mu)$$

we get an identification

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{F : \widetilde{\Omega} \to \boldsymbol{W} \otimes_{\mathbb{Q}} E :\ F \text{ continuous; and } F((\zeta,g),\mu) \in \boldsymbol{r}(g)\boldsymbol{W} = {}^{\zeta^{-1}}\boldsymbol{W};\\ \text{and } F((i_{(\phi,b)}(\gamma),\mathrm{conj}_\gamma)x) = \boldsymbol{r}(b^{-1}\gamma b)F(x)\ \forall\gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\},\end{aligned}$$

where now $\widetilde{\Omega}$ is the preimage of $\pi^{-1}_{(\phi,b)}\Omega \subset {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}}\backslash(\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{\phi G})$ in the space $\widetilde{G}_\psi(\mathbb{A}^\infty)/\overline{Z(G)^1(\mathbb{Q})}U \times Y(C)_{\phi G}$. Similarly using ${}^{(\delta,(\eta,h))}(\phi,b)$ we have

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}(\Omega) = \{F' : ((\eta,h),\mathrm{conj}_\delta)\widetilde{\Omega} \to \boldsymbol{W} \otimes_{\mathbb{Q}} E :\ F' \text{ continuous; and}\\ F'((\zeta,g),\mu) \in \boldsymbol{r}(g)\boldsymbol{W} = {}^{\zeta^{-1}}\boldsymbol{W};\\ \text{and } F'((i_{(\phi,b)}(\gamma),\mathrm{conj}_\gamma)x) = \boldsymbol{r}(b^{-1}\gamma b)F'(x)\ \forall\gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\}.\end{aligned}$$

The canonical identification of these two sets is via:

$$F'(y) = \boldsymbol{r}(h^\infty)F(((\eta,h),\mathrm{conj}_\delta)^{-1}y).$$

Because $\widetilde{G}_\psi(\mathbb{A}^\infty)/\overline{Z(G)^1(\mathbb{Q})}U \times Y(C)_{\phi G}$ is locally connected the lemma follows. $\square$

Now suppose that $(\phi, b, W, r, \alpha) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, C, \boldsymbol{W}, \boldsymbol{r})$. We define a locally constant sheaf $\mathcal{W}_{(\phi,b,W,r,\alpha)}$ on $\mathrm{Sh}(G, \psi, C, U)$ by

$$\begin{aligned}\mathcal{W}_{(\phi,b,W,r,\alpha)}(\Omega) = \{F : \widetilde{\Omega} \to W \otimes_{\mathbb{Q}} E :\ & F \text{ locally constant; and } F((\zeta, g), \mu) \in {}^{\zeta^{-1}}W; \\ & \text{and } F((i_{(\phi,b)}(\gamma), \mathrm{conj}_\gamma)x) = r(\gamma)F(x)\ \forall \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\},\end{aligned}$$

where $\widetilde{\Omega}$ is the preimage of $\pi^{-1}_{(\phi,b)}\Omega \subset {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_{\phi G})$ in the space $\widetilde{G}_\psi(\mathbb{A}^\infty)/\overline{Z(G)^1(\mathbb{Q})}U \times Y(C)_{\phi G}$. The map $F \mapsto \alpha \circ F$ gives an isomorphism

$$\widetilde{\alpha}_{(\phi,b,W,r,\alpha)} : \mathcal{W}_{(\phi,b,W,r,\alpha)} \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} \boldsymbol{\mathcal{W}}^{an}_{\boldsymbol{r}}.$$

We must analyze how this construction depends on (ϕ, b, W, r, α). We claim that if $(\phi', b', W', r', \alpha') \sim (\phi, b, W, r, z\alpha)$, then there is a (necessarilly unique) isomorphism $\mathcal{W}_{(\phi,b,W,r,\alpha)} \cong \mathcal{W}_{(\phi',b',W',r',\alpha')}$ such that

$$\begin{array}{ccc} \mathcal{W}_{(\phi,b,W,r,\alpha)} & \cong & \mathcal{W}_{(\phi',b',W',r',\alpha')} \\ \widetilde{\alpha}_{(\phi,b,W,r,\alpha)} \downarrow & & \downarrow \widetilde{\alpha}_{(\phi',b',W',r',\alpha')} \\ \boldsymbol{\mathcal{W}}_{\boldsymbol{r}} & \underset{z}{\longrightarrow} & \boldsymbol{\mathcal{W}}_{\boldsymbol{r}} \end{array}$$

commutes. Indeed if the equivalence of $(\phi, b, W, r, z\alpha)$ and $(\phi', b', W', r', \alpha')$ are equivalent, and this equivalence is instanced by $((\delta, (\eta, h)), \beta)$ then we define an isomorphism between $\mathcal{W}_{(\phi,b,W,r,\alpha)}$ and $\mathcal{W}_{(\phi',b',W',r',\alpha')}$ by identifying

$$\begin{aligned}\mathcal{W}_{(\phi,b,W,r,\alpha)}(\Omega) = \{F : \widetilde{\Omega} \to W \otimes_{\mathbb{Q}} E :\ & F \text{ locally constant; and } F((\zeta, g), \mu) \in {}^{\zeta^{-1}}W; \\ & \text{and } F((i_{(\phi,b)}(\gamma), \mathrm{conj}_\gamma)x) = r(\gamma)F(x)\ \forall \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\}\end{aligned}$$

with

$$\begin{aligned}\mathcal{W}_{(\phi',b',W',r',\alpha')}(\Omega) = \{F' : ((\eta, h), \mathrm{conj}_\delta)\widetilde{\Omega} \to W' \otimes_{\mathbb{Q}} E :\ & F' \text{ locally constant; and} \\ & F'((\zeta, g), \mu) \in {}^{\zeta^{-1}}W'; \text{ and} \\ F'((((\eta, h)i_{(\phi,b)}(\delta^{-1}\gamma\delta)(\eta, h)^{-1}, \mathrm{conj}_\gamma)((\eta, h), \mathrm{conj}_\delta)x) = & r'(\gamma)F'(((\eta, h), \mathrm{conj}_\delta)x) \\ & \forall \gamma \in {}^{\delta\phi\eta}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\},\end{aligned}$$

i.e.

$$\begin{aligned}\mathcal{W}_{(\phi',b',W',r',\alpha')}(\Omega) = \{F' : ((\eta, h), \mathrm{conj}_\delta)\widetilde{\Omega} \to W' \otimes_{\mathbb{Q}} E :\ & F' \text{ locally constant; and} \\ & F'((\zeta, g), \mu) \in {}^{\zeta^{-1}}W'; \text{ and} \\ F'((((\eta, h), \mathrm{conj}_\delta)(i_{(\phi,b)}(\gamma), \mathrm{conj}_\gamma)x) = & r'(\delta\gamma\delta^{-1})F'(((\eta, h), \mathrm{conj}_\delta)x) \\ & \forall \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\},\end{aligned}$$

via

$$F'(((\eta, h), \mathrm{conj}_\delta)x) = \beta \circ F(x).$$

Using lemma 4.7, it is easy to check that this identification makes the desired diagram commute.

Associated to $\mathcal{W}_{\boldsymbol{r}}$ we have a locally free sheaf of $\mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}$-modules with connection

$$(\mathcal{W}_{\boldsymbol{r}} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}, 1 \otimes d).$$

Here $\mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}$ denotes the sheaf of analytic functions on $\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})$. Similarly for $\mathcal{W}_{(\phi,b,W,r,\alpha)}$. We have

$$(\mathcal{W}_{(\phi,b,W,r,\alpha)} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})})(\Omega) = \{F : \widetilde{\Omega} \to (W \otimes_{\mathbb{Q}} E) \otimes_{\mathbb{Q}} \mathbb{C} :\ F \text{ holomorphic; and } F((\zeta,g),\mu) \in {}^{\zeta^{-1}}W \otimes_{\mathbb{Q}} \mathbb{C};\ \text{and } F((i_{(\phi,b)}(\gamma), \mathrm{conj}_{\gamma})x) = r(\gamma)F(x)\ \forall \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\},$$

where $\widetilde{\Omega}$ is the preimage of $\pi^{-1}_{(\phi,b)}\Omega \subset {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}} \backslash (\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/U \times Y(C)_{\phi G})$ in the space $\widetilde{G}_{\psi}(\mathbb{A}^{\infty})/\overline{Z(G)^1(\mathbb{Q})}U \times Y(C)_{\phi G}$.

If $\mu \in Y(C)_{\phi G}$ and $(\phi,b,W,r,\alpha) \in \mathrm{Label}_{\mathfrak{a}}(G,\psi,C,\boldsymbol{W},\boldsymbol{r})$ and $\zeta \in Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G))$ we define a filtration $\mathrm{Fil}^{\,i}_{(\phi,b,W,r,\alpha),\mu}$ on ${}^{\zeta}W \otimes_{\mathbb{Q}} \mathbb{C}$ by setting

$$\mathrm{Fil}^{\,i}_{(\phi,b,W,r,\alpha),\mu}{}^{\zeta}W \otimes_{\mathbb{Q}} \mathbb{C}$$

to be the sum of the j weight spaces for $r \circ \mu$ for $j \leq -i$. If $((\delta,(\eta,h)),\beta)$ establishes an equivalence between (ϕ,b,W,r,α) and (ϕ',b',W',r',α'), then

$$\beta \mathrm{Fil}^{\,i}_{(\phi,b,W,r,\alpha),\mu}{}^{\zeta}W \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{Fil}^{\,i}_{(\phi',b',W',r',\alpha'),\mathrm{conj}_{\delta}\circ\mu}{}^{\zeta}W \otimes_{\mathbb{Q}} \mathbb{C}.$$

We define

$$\mathrm{Fil}^{\,i}(\mathcal{W}_{(\phi,b,W,r,\alpha)} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})})(\Omega) = \{F : \widetilde{\Omega} \to (W \otimes_{\mathbb{Q}} E) \otimes_{\mathbb{Q}} \mathbb{C} :\ F \text{ holomorphic; and } F((\zeta,g),\mu) \in \mathrm{Fil}^{\,i}_{(\phi,b,W,r,\alpha),\mu}{}^{\zeta^{-1}}W \otimes_{\mathbb{Q}} \mathbb{C};\ \text{and } F((i_{(\phi,b)}(\gamma), \mathrm{conj}_{\gamma})x) = r(\gamma)F(x)\ \forall \gamma \in {}^{\phi}G(E)^{\mathbb{Q}}_{\mathbb{R}},\ x \in \widetilde{\Omega}\}.$$

We see immediately that this filtration is preserved under the canonical identification of $\mathcal{W}_{(\phi,b,W,r,\alpha)}$ and $\mathcal{W}_{(\phi',b',W',r',\alpha')}$ and so we get a canonical decreasing filtration $\mathrm{Fil}^{\,i}(\mathcal{W}_{\boldsymbol{r},\lambda} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})})$ on $\mathcal{W}_{\boldsymbol{r},\lambda} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathrm{Sh}(G,\psi,C,U)(\mathbb{C})}$. We claim that this defines a variation of rational Hodge structures, with the associated variation of real Hodge structures polarizable. Indeed this question is local and so reduces to the corresponding question for $Y(C)_{\phi G}$, where it is part of proposition 1.1.14 of [D2]. Finally, if $z \in Z(\boldsymbol{r})$, then the identifications of $\mathcal{W}_{\boldsymbol{r},\lambda}$ and $\mathcal{W}_{\boldsymbol{r},z\lambda}$ preserve these filtrations. Thus we have completed the advertised construction.

5. Relationship between rational Shimura varieties and some moduli problems for abelian varieties

At the suggestion of Pol van Hoften we explain the connection between certain moduli problems for abelian varieties with polarizations and endomorphisms considered by Kottwitz and special cases of our rational Shimura varieties.

Following Kottwitz [K1] considers tuples (which we will call 'PEL data')

$$(B, *, V, (\ ,\ ), h)$$

where:

- B is a finite dimensional simple $\mathbb{Q}$-algebra.
- $*$ is a positive involution on B.
- V is a finitely generated left B-module.
- $(\ ,\ ) : V \times V \to \mathbb{Q}$ is a non-degenerate alternating form such that $(bx, y) = (x, b^* y)$ for all $b \in B$ and $x, y \in V$.
- $h : \mathbb{C} \to \mathrm{End}_B(V) \otimes_{\mathbb{Q}} \mathbb{R}$ is a map of $\mathbb{R}$-algebras such that
  - $(h(z)x, y) = (x, h({}^c z)y)$ for all $z \in \mathbb{C}$ and $x, y \in V \otimes_{\mathbb{Q}} \mathbb{R}$
  - and the (necessarily) symmetric $\mathbb{R}$-bilinear form $(\ , h(i)\ )$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite.

Then $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_c$, where $h(z) \otimes 1 = 1 \otimes z$ on V_1 and $h(z) \otimes 1 = 1 \otimes {}^c z$ on V_c for all $z \in \mathbb{C}$. Note that V_1 and V_c are isotropic, and dual to each other under $(\ ,\ )$. Kottwitz defines the 'reflex field' $L(B, *, V, (\ ,\ ), h) \subset \mathbb{C}$ to be the field of definition of the isomorphism class of the B representation V_1. He also defines a not necessarily connected reductive group $G = G_{(B,*,V,(\ ,\ ),h)}$ over $\mathbb{Q}$ of B-linear automorphisms of V which preserve $(\ ,\ )$ up to a scalar multiple. Thus there is a character $\nu : G \to \mathbb{G}_m$ defined over $\mathbb{Q}$ such that $(gx, gy) = \nu(g)(x, y)$. We can define $\mathrm{wt}, \mu_h \in X_*(G)(\mathbb{C})$ by requiring that $\mathrm{wt}(z)$ acts by z on V, while $\mu_h(z)$ acts by z on V_1 and by 1 on V_c. Then wt is defined over $\mathbb{Q}$ and the geometric conjugacy class $[\mu_h]_G$ is defined over $L(B, *, V, (\ ,\ ), h)$. (To see this suppose $\sigma \in \mathrm{Aut}(\mathbb{C}/L(B, *, V, (\ ,\ ), h))$. Then ${}^\sigma \mu_h$ acts as z on ${}^{1\otimes\sigma} V_1$ and as 1 on ${}^{1\otimes\sigma} V_c$. By definition there is a $B \otimes_{\mathbb{Q}} \mathbb{C}$-linear isomorphism $f : V_1 \xrightarrow{\sim} {}^{1\otimes\sigma} V_1$. Then $f \oplus (f^\vee)^{-1} : V \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{C}$ is $B \otimes_{\mathbb{Q}} \mathbb{C}$-linear, preserves $(\ ,\ )$ and takes V_1 to ${}^{1\otimes\sigma} V_1$ and V_c to ${}^{1\otimes\sigma} V_c$. Thus $f \oplus (f^\vee)^{-1} \in G(\mathbb{C})$ and ${}^\sigma \mu_h = \mathrm{conj}_{f \oplus (f^\vee)^{-1}} \circ \mu_h$, as desired.)

If $U \subset G(\mathbb{A}^\infty)$ Kottwitz further considers the moduli problem on locally noetherian $L(B, *, V, (\ ,\ ), h)$-schemes, which sends S to the set of equivalence classes of 4-tuples $(A, \lambda, i, [\eta])$, where

- A/S is an abelian scheme of dimension $(1/2) \dim_{\mathbb{Q}} V$,
- $\lambda : A \to A^\vee$ is a polarization,
- $i : B \hookrightarrow \mathrm{End}(A/S) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

  $$\mathrm{tr}\,(i(b)|_{\mathrm{Lie}\,A}) = \mathrm{tr}\,((b \otimes 1)|_{V_1}) \in L(B, *, V, (\ ,\ ), h)$$

  for all $b \in B$,

- and $[\eta]$ is a U-level structure on (A, λ, i).

If S is connected and s is a geometric point of S, then by a U-level structure on (A, λ, i) we mean a $\pi_1(S, s)$-invariant U-orbit $[\eta]$ of $(B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)$-linear isomorphisms η from $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ to the $\mathbb{A}^\infty$-Tate module of A_s which takes $(\ ,\ )$ to an $(\mathbb{A}^\infty)^\times$ multiple of the λ-Weil pairing. (Note that the pairing $(\ ,\ )$ is valued in $\mathbb{A}^\infty$, while the λ-Weil pairing is valued in $\mathbb{A}^\infty(1)$, but since we are only requiring one pairing to match with the other up to $(\mathbb{A}^\infty)^\times$-multiples, it doesn't matter how we identify $\mathbb{A}^\infty$ and $\mathbb{A}^\infty(1)$.) This is canonically independent of the choice of geometric point s. If S is simply locally noetherian, then it is the disjoint union of its connected components and by a U-level structure on $(A, \lambda, i)/S$, we mean the choice of one on each connected component of S.

We consider two 4-tuples $(A, \lambda, i, [\eta])$ and $(A', \lambda', i', [\eta'])$ equivalent if there is a B-linear isogeny $\beta : A \to A'$ such that $\beta_*[\eta] = [\eta']$ and $\lambda' \circ \beta = \gamma \circ \beta^\vee \circ \lambda$ for some $\gamma \in \mathbb{Q}_{>0}^\times$.

Kottwitz explains that if U is sufficiently small then this moduli problem is represented by a smooth quasi-projective scheme $\pi : \mathcal{A}_U \to S_U = S(B, *, V, (\ ,\ ), h)_U$ over the field $L(B, *, V, (\ ,\ ), h)$. If $g \in G(\mathbb{A}^\infty)$ and $gU'g^{-1} \subset U$ then there is a finite etale map $g : S_{U'} \to S_U$ coming from the U structure on $(A^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, i^{\mathrm{univ}})/S_{U'}$ given as the U orbit of $\eta^{\mathrm{univ}} \circ g^{-1}$. (Kottwitz actually works over a localization of $\mathcal{O}_{L(B,*,V,(\ ,\ ),h)}$, and this required him to replace our 'trace condition' with a 'determinant condition'. However his stronger results easily implies the results recalled here.)

If $(A, \lambda, i, [\eta])$ is a tuple representing a point of $S_U(\mathbb{C})$ then we may associate to it a tuple $(H^1(A(\mathbb{C}), \mathbb{Q}), (\ ,\ )_\lambda, \mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}, [\eta^{\vee,-1}])$, where

- $H^1(A(\mathbb{C}), \mathbb{Q})$ is a finitely generated right B-module;
- $(\ ,\ )_\lambda$ is the λ-Weil pairing
$$H^1(A(\mathbb{C}), \mathbb{Q}) \times H^1(A(\mathbb{C}), \mathbb{Q}) \longrightarrow (2\pi i)^{-1}\mathbb{Q}$$
which is a non-degenerate alternating pairing satisfying $(xb, y)_\lambda = (x, yb^*)_\lambda)$ for all $b \in B$ and $x, y \in H$;
- $\mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is a maximal isotropic B-invariant subspace such that $(\mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}) \cap {}^c(\mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}) = (0)$ and there is an isomorphism of $B \otimes_{\mathbb{Q}} \mathbb{C}$ modules $V_1^\vee \cong \mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$;
- if $(\ ,\ )_{\lambda, \mathrm{gr}^{-1}}^\vee$ is the $\mathbb{R}$-valued, $\mathbb{R}$-bilinear pairing on
$$(\mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C})^\vee = (H^1(A(\mathbb{C}), \mathbb{Q})^\vee \otimes_{\mathbb{Q}} \mathbb{C})/\mathrm{Fil}^1 (H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C})^\perp$$
induced by $H^1(A(\mathbb{C}), \mathbb{Q})^\vee \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} (\mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q})_{\mathbb{C}})^\vee$ and $((2\pi i)(\ ,\ )_\lambda)^\vee$, then the necessarily symmetric pairing $(\ , i\ )_{\lambda, \mathrm{gr}^{-1}}^\vee$ is positive definite;
- $[\eta^{\vee,-1}]$ is a U-orbit of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-linear isomorphisms $\eta^{\vee,-1} : V^\vee \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} \mathrm{Fil}^1 H^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ taking $(\ ,\ )^\vee$ to $(\ ,\ )_\lambda$ for some $\mathbb{A}^\infty$ linear identification $\mathbb{A}^\infty \xrightarrow{\sim} (2\pi i)^{-1}\mathbb{A}^\infty$.

We will write $G_c = {}^{(\mu_h{}^c\mu_h)^{-1},\mu_h^{-1}(-1)}G$, which is compact mod centre, and let C be the G_c-conjugacy class of μ_h. Then $Y(C)_G = [\mu_h]_{G(\mathbb{R})}$. We will also write Gr for the Grassmannian of maximal B-invariant isotropic subspaces $W \subset V^\vee$ such that $W \cong V_1^\vee$ as B-modules. Then $\mathrm{Gr} = G/\mathrm{Stab}_G((V^\vee \otimes_{\mathbb{Q}} \mathbb{C})^{\mu_h(-1)=-1}) = G/P_{\mu_h}^-$. (Note that $\mu_h(-1) = -1$ is equivalent to $\mu_h(z) = z^{-1}$ for all $z \in \mathbb{Z}^\times$.) Thus there is a natural embedding

$$Y(C)_G \hookrightarrow \mathrm{Gr}(\mathbb{C})$$

sending μ to $(V^\vee \otimes_{\mathbb{Q}} \mathbb{C})^{\mu(-1)=-1}$. The map is a biholomorphic isomorphism of $Y(C)_G$ with an open subset of $\mathrm{Gr}(\mathbb{C})$.

The centre $Z(B)$ of B is a CM (possibly totally real) field with maximum totally real subfield $Z(B)^+$. Set $d = [Z(B)^+ : \mathbb{Q}]$. We have $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_{n\times n}(\mathbb{C})^d$ or $M_{n\times n}(\mathbb{R})^d$ or $M_{n/2\times n/2}(\mathbb{H})^d$, where $\mathbb{H}$ denotes the Hamiltonian quaternions. These are referred to as cases A, C and D respectively. *For the rest of this section we assume we are in case A or C, but not D.* This ensures that G is connected. (See section 5 of [K1].)

Choose a finite Galois extension $E/\mathbb{Q}$ which is acceptable for E, and an element $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$. Also choose an infinite place w_∞ of E and a section $s : \{1, c_{w_\infty}\}\backslash \mathrm{Gal}\,(E/\mathbb{Q}) \to \mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\infty,\mathfrak{a}}$. Define $\psi_{w_\infty} \in Z^1_{\mathrm{alg}}(W_{E_{w_\infty}/\mathbb{R},\mathfrak{a}}, G(E_{w_\infty}))$ by

$$\psi_{w_\infty}|_{E_{w_\infty}^\times} = \mathrm{wt}$$

and

$$\psi_{w_\infty}(j_{w_\infty}) = \mu_h(-1).$$

Then define $\psi_\infty \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\infty,\mathfrak{a}}, G(E_\infty))$ by

$$\psi_\infty(\sigma) = \prod_{\tau \in \{1,c_{w_\infty}\}\backslash \mathrm{Gal}\,(E/\mathbb{Q})} s(\tau)^{-1}\psi_{w_\infty}(s(\tau)\sigma s(\tau\sigma)^{-1})$$

and

$$\psi = (1, \psi_\infty) \in Z^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{loc}}(E/\mathbb{Q})_{\mathfrak{a}}, G(\mathbb{A}_E)).$$

Then

$$(G, \psi, [\mu]_G, U) \in \mathrm{RSD}(E, \mathfrak{a}; L(B, *, V, (\ ,\ ), h)).$$

Note that $(1,1) \in \mathrm{Label}_{\mathfrak{a}}(G, \psi, [\mu]_G)$. Write $\boldsymbol{r}$ for the representation of ${}^\psi G$ on $V^\vee \otimes_{\mathbb{Q}} \mathbb{A}^\infty$. It is rationalizable. (By the representation r of G on $V^\vee$.)

In the rest of this section we will prove the following theorem.

Theorem 5.1. *Suppose that $(B, *, V, (\ ,\ ), h)$ is PEL data of type A or C. Let $G/\mathbb{Q}$ be the reductive group defined above and $U \subset G(\mathbb{A}^\infty)$ be a sufficiently small open compact subgroup. Choose $E/\mathbb{Q}$ acceptable for G and $\mathfrak{a}^+ \in \mathcal{H}(E/\mathbb{Q})^+$. Let*

$$(G, \psi, [\mu]_G, U) \in \mathrm{RSD}(E, \mathfrak{a}; \mathbb{C})$$

be as defined above. Then there is an isomorphism

$$S(B, *, V, (\ ,\ ), h)_U \xrightarrow{\sim} \mathrm{Sh}(G, \psi, [\mu_h^{-1}], U)$$

(of varieties over $\mathbb{C}$). If (A,λ,i,η) maps to y then there is an isomorphism of Hodge structures

$$(H^1(A,\mathbb{Q}),\mathrm{Fil}^1\mathrm{Fil}^1H^1(A(\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C})\cong(\mathcal{W}_{\boldsymbol{r},y},\mathrm{Fil}^1\mathcal{W}_{\boldsymbol{r},y}\otimes_{\mathbb{Q}}\mathbb{C}).$$

It should be routine to extend this theorem to an isomorphism of varieties over $L(B,*,V,(\ ,\ ),h)$ by keeping track of what happens at CM points, but we have not carried out this exercise. The strategy of the proof will first be to exhibit a bijection of $\mathbb{C}$-points, then show the map is holomorphic and deduce from [B2] that it is isomorphism of varieties.

Consider the set $\mathrm{LAD}\,(B,*,V,(\ ,\ ),h)_U$ of tuples

$$(H,(\ ,\ )_H,\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_H])$$

where

- H is a finitely generated right B-module;
- $(\ ,\ )_H : H\times H\to(2\pi i)^{-1}\mathbb{Q}$ is a non-degenerate alternating form such that $(xb,y)_H=(x,yb^*)_H$ for all $b\in B$ and $x,y\in H$;
- $\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})\subset(H\otimes_{\mathbb{Q}}\mathbb{C})$ is a maximal isotropic B-invariant subspace such that $\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})\cap{}^{1\otimes c}\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})=(0)$ and such that there is an isomorphism of $B\otimes_{\mathbb{Q}}\mathbb{C}$-modules $V_1^\vee\cong\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})$;
- if $(\ ,\ )^\vee_{H,\mathrm{gr}^{-1}}$ is the $\mathbb{R}$-valued, $\mathbb{R}$-bilinear pairing on $(\mathrm{Fil}^1H_{\mathbb{C}})^\vee=(H^\vee\otimes_{\mathbb{Q}}\mathbb{C})/\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})^\perp$ induced by $H^\vee\otimes_{\mathbb{Q}}\mathbb{R}\xrightarrow{\sim}(\mathrm{Fil}^1H_{\mathbb{C}})^\vee$ and $((2\pi i)(\ ,\ )_H)^\vee$, then the necessarily symmetric pairing $(\ ,i\ )^\vee_{H,\mathrm{gr}^{-1}}$ is positive definite;
- $[\eta_H]$ is a U-orbit of $B\otimes_{\mathbb{Q}}\mathbb{A}^\infty$-linear isomorphisms $\eta_H:V^\vee\otimes_{\mathbb{Q}}\mathbb{A}^\infty\xrightarrow{\sim}H\otimes_{\mathbb{Q}}\mathbb{A}^\infty$ taking $(\ ,\ )^\vee$ to $(\ ,\ )_H$ for some $\mathbb{A}^\infty$ linear identification $\mathbb{A}^\infty\xrightarrow{\sim}(2\pi i)^{-1}\mathbb{A}^\infty$.

We call two such triples $(H,(\ ,\ )_H,\mathrm{Fil}^0(H\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_H])$ and $(H',(\ ,\ )_{H'},\mathrm{Fil}^0(H'\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_{H'}])$ *equivalent* if there is an isomorphism of B-modules

$$\beta:H\xrightarrow{\sim}H'$$

which takes $(\ ,\ )_H$ to a $\mathbb{Q}^\times_{>0}$-multiple of $(\ ,\ )_{H'}$ and $\mathrm{Fil}^0(H\otimes_{\mathbb{Q}}\mathbb{C})$ to $\mathrm{Fil}^0(H'\otimes_{\mathbb{Q}}\mathbb{C})$ and $[\eta_H]$ to $[\eta_{H'}]$. We write $(H,(\ ,\ )_H,\mathrm{Fil}^0(H\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_H])\sim(H',(\ ,\ )_{H'},\mathrm{Fil}^0(H'\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_{H'}])$.

Lemma 5.2. *The map*

$$(A,\lambda,i,[\eta])\longmapsto(H^1(A(\mathbb{C}),\mathbb{Q}),(\ ,\ )_\lambda,\mathrm{Fil}^1H^1(A(\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C},[\eta^{\vee,-1}])$$

induces a bijection

$$S(B,*,V,(\ ,\ ),h)_U(\mathbb{C})\xrightarrow{\sim}\mathrm{LAD}\,(B,*,V,(\ ,\ ),h)_U/\sim.$$

Proof: We must first check that the given map is well defined. If $\beta:A\to A'$ is an isogeny giving an equivalence of $(A,i,\lambda,[\eta])$ and $(A',i',\lambda',[\eta'])$, then β induces an isomorphism $H^1(A(\mathbb{C}),\mathbb{Q}) \to H^1(A'(\mathbb{C}),\mathbb{Q})$ giving an equivalence

$$\begin{array}{cl} & (H^1(A'(\mathbb{C}),\mathbb{Q}),(\ ,\ )_\lambda,\mathrm{Fil}^1 H^1(A'(\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C},[\eta^{\prime,\vee,-1}]) \\ \sim & (H^1(A(\mathbb{C}),\mathbb{Q}),(\ ,\ )_\lambda,\to H^1(A(\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C},[\eta^{\vee,-1}]). \end{array}$$

We can define a map in the other direction by choosing a lattice $\Lambda \subset H^\vee$ on which $((2\pi i)^{-1}(\ ,\ )_H)^\vee$ is $\mathbb{Z}$-valued, and sending $(H,(\ ,\ )_H,\mathrm{Fil}^0(H\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_H])$ to $[(A,i,\lambda,[\eta])]$, where

- $A(\mathbb{C}) = (H^\vee \otimes_{\mathbb{Q}} \mathbb{C})/(\Lambda + (\mathrm{Fil}^1 H \otimes_{\mathbb{Q}} \mathbb{C})^\perp)$,
- $(\ ,\ )_H$ is the λ-Weil pairing,
- and $[\eta]$ is induced by $[\eta_H]$ and the identification of the Tate module of A with $H^\vee \otimes_{\mathbb{Q}} \mathbb{A}^\infty$.

The equivalence class of $(A,i,\lambda,[\eta])$ does not depend on the choice of Λ, and only depends on the equivalence class of $(H,(\ ,\ )_H,\mathrm{Fil}^0(H\otimes_{\mathbb{Q}}\mathbb{C}),[\eta_H])$.

These two maps are easily checked to be two sided inverses to each other. □

We want to provide a reformulation of the data $\mathrm{LAD}\,(B,*,V,(\ ,\ ),h)_U$. First we may replace $(\ ,\ )_H$ with $2\pi i(\ ,\ )_H$, with the obvious modification of the conditions. Secondly we may replace the choice of $\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})$ with a homomorphism of $\mathbb{R}$-algebras $h_H : \mathbb{C} \to \mathrm{End}_B(H\otimes_{\mathbb{Q}}\mathbb{R})$ such that $h_H(z)$ acts as z on $\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})$ and as ${}^c z$ on ${}^c\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C})$. Then h_H satisfies the following conditions:

- $(x,h_H(z)y)_H = (h_H({}^c z)x,y)_H$,
- $V_1^\vee \cong (H\otimes_{\mathbb{Q}}\mathbb{C})^{h(i)=i}$ as $B\otimes_{\mathbb{Q}}\mathbb{C}$-modules,
- $(\ ,h_H(i)\ )_H$, which is necessarily symmetric, is positive definite on $H\otimes_{\mathbb{Q}}\mathbb{R}$.

Conversely if h_H satisfies these conditions then $\mathrm{Fil}^1(H\otimes_{\mathbb{Q}}\mathbb{C}) = (H\otimes_{\mathbb{Q}}\mathbb{C})^{h_H(i)=i}$ will satisfy the conditions defining an element of $\mathrm{LAD}\,(B,*,V,(\ ,\ ),h)_U$. Moreover, by lemma 4.2 of [?], these three conditions are equivalent to

- there is a $B\otimes_{\mathbb{Q}}\mathbb{R}$-linear isomorphism $V^\vee\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} H\otimes_{\mathbb{Q}}\mathbb{R}$ which takes $(\ ,\ )^\vee$ to an $\mathbb{R}^\times_{>0}$ multiple of $(\ ,\ )_H$ and takes $h^\vee$ to h_H.

Thus we may think of $\mathrm{LAD}\,(B,*,V,(\ ,\ ),h)_U$ as the set of tuples $(H,(\ ,\ )_H,h_H,[\eta_H])$ where:

- H is a finitely generated right B-module;
- $(\ ,\ )_H : H\times H \to \mathbb{Q}$ is a non-degenerate alternating form such that $(xb,y)_H = (x,yb^*)_H$ for all $b\in B$ and $x,y\in H$;
- $h_H : \mathbb{C} \to \mathrm{End}_{B\otimes_{\mathbb{Q}}\mathbb{R}}(H\otimes_{\mathbb{Q}}\mathbb{R})$ such that there exists a $B\otimes_{\mathbb{Q}}\mathbb{R}$-linear isomorphism $V^\vee\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} H\otimes_{\mathbb{Q}}\mathbb{R}$ which takes $(\ ,\ )^\vee$ to an $\mathbb{R}^\times_{>0}$ multiple of $(\ ,\ )_H$ and takes $h^\vee$ to h_H;
- $[\eta_H]$ is a U-orbit of $B\otimes_{\mathbb{Q}}\mathbb{A}^\infty$-linear isomorphisms $\eta_H : V^\vee\otimes_{\mathbb{Q}}\mathbb{A}^\infty \xrightarrow{\sim} H\otimes_{\mathbb{Q}}\mathbb{A}^\infty$ taking $(\ ,\ )^\vee$ to a $(\mathbb{A}^\infty)^\times$-multiple of $(\ ,\ )_H$.

Two such tuples $(H, (\ ,\ )_H, h_H, [\eta_H])$ and $(H', (\ ,\ )_{H'}, h_{H'}, [\eta_{H'}])$ are equivalent if and only if there is an isomorphism of B-modules

$$\beta : H \xrightarrow{\sim} H'$$

which takes $(\ ,\ )_H$ to a $\mathbb{Q}^\times_{>0}$-multiple of $(\ ,\ )_{H'}$ and h_H to $h_{H'}$ and $[\eta_H]$ to $[\eta_{H'}]$.

If we write G_H for the group of B-linear automorphisms of H which preserve $(\ ,\ )_H$ up to scalar multiples, then we may replace h_H by the induced map $\mathrm{RS}^{\mathbb{C}}_{\mathbb{R}}\mathbb{G}_m \to G_H$ over $\mathbb{R}$ which it induces. (This does not enlarge the collection of tuples we are considering because the weights of h_H on $V^\vee$ must be $(1,0)$ and $(0,1)$.) Then $h_{H,\mathbb{C}} = (\mu^{-1}, {}^c\mu^{-1})$ where $\mu \in X_*(G_H)(\mathbb{C})$ commutes with ${}^c\mu$. Thus we may also think of $\mathrm{LAD}\,(B, *, V, (\ ,\ ), h)_U$ as the set of tuples $(H, (\ ,\ )_H, \mu_H, [\eta_H])$ where:

- H is a finitely generated right B-module;
- $(\ ,\ )_H : H \times H \to \mathbb{Q}$ is a non-degenerate alternating form such that $(xb, y)_H = (x, yb^*)_H$ for all $b \in B$ and $x, y \in H$;
- $\mu_H \in X_*(G_H)(\mathbb{C})$ such that there exists a $B \otimes_{\mathbb{Q}} \mathbb{R}$-linear isomorphism $V^\vee \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H \otimes_{\mathbb{Q}} \mathbb{R}$ which takes $(\ ,\ )^\vee$ to an $\mathbb{R}^\times_{>0}$ multiple of $(\ ,\ )_H$ and takes μ_h to μ_H;
- $[\eta_H]$ is a U-orbit of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-linear isomorphisms $\eta_H : V^\vee \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} H \otimes_{\mathbb{Q}} \mathbb{A}^\infty$ taking $(\ ,\ )^\vee$ to a $(\mathbb{A}^\infty)^\times$-multiple of $(\ ,\ )_H$.

Two such triples $(H, (\ ,\ )_H, \mu_H, [\eta_H])$ and $(H', (\ ,\ )_{H'}, \mu_{H'}, [\eta_{H'}])$ are equivalent if and only if there is an isomorphism of B-modules

$$\beta : H \xrightarrow{\sim} H'$$

which takes $(\ ,\ )_H$ to a $\mathbb{Q}^\times_{>0}$-multiple of $(\ ,\ )_{H'}$ and μ_H to $\mu_{H'}$ and $[\eta_H]$ to $\eta_{h'}]$.

Lemma 5.3. *Suppose we are in case A or C. Then* $\mathrm{LAD}\,(B, *, V, (\ ,\ ), h)_U$ *is in bijection with* $\mathrm{Sh}(G, \psi, C, U)(\mathbb{C})$ *via the map which sends* $\pi_{(1,1)}[((\zeta, g), \mu)]$ *to the tuple* $[({}^{\zeta^{-1}}(V^\vee), {}^{\zeta^{-1}}(\ ,\ )^\vee, \mu, g^\infty U)]$, *where* ${}^{\zeta^{-1}}(\ ,\ )^\vee$ *is obtained from* $(\ ,\ )^\vee$ *and any identification of* ${}^{\nu\circ\zeta^{-1}}\mathbb{Q}$ *with* $\mathbb{Q}$.

Proof: Using $\pi_{(1,1)}$ we may replace $\mathrm{Sh}(G, \psi, C, U)(\mathbb{C})$ with $G(E)^{\mathbb{Q}}_{\mathbb{R}} \backslash (\widetilde{G}_\psi(\mathbb{A}^\infty)/U \times Y(C)_G)$. The given map is easily checked to be well defined. (One needs to notice that, by lemma 4.1, ${}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{R} = zV^v \otimes_{\mathbb{Q}} \mathbb{R}$ for some $z \in Z(G)(\mathbb{C})$. If $\mu = \mathrm{conj}_k \circ \mu_h$ with $k \in G(\mathbb{R})$, then $zk : V^\vee \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} {}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{R}$ takes $(\ ,\ )^\vee$ to an $\mathbb{R}^\times$-multiple of ${}^{\zeta^{-1}}(\ ,\ )$ and μ_h to μ.) To see the map is injective suppose that $((\zeta, g), \mu)$ and $((\zeta', g'), \mu') \in \widetilde{G}_\psi(\mathbb{A}^\infty) \times Y(C)_G$ with

$$({}^{\zeta^{-1}}(V^\vee), {}^{\zeta^{-1}}(\ ,\ )^\vee, \mu, g^\infty U) \sim ({}^{\zeta',-1}(V^\vee), {}^{\zeta',-1}(\ ,\ )^\vee, \mu', g'^{,\infty} U).$$

Then there must be $\beta \in G(E)$ with $\beta^{\zeta^{-1}}V^\vee = {}^{\zeta',-1}V^\vee$, so that $\zeta'\zeta^{-1}(\sigma) = \beta^{-1\sigma}\beta$ for all $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q})$. In particular $\beta \in G(E)^{\mathbb{Q}}$. Because the restriction to $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\zeta'\zeta^{-1} \in H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G))$ is trivial, we see that in fact $\beta \in G(E)^{\mathbb{Q}}_{\mathbb{R}}$. Moreover $\mu' = \mathrm{conj}_\beta \circ \mu$.

Finally we need to check surjectivity. So suppose that $[(H, (\ ,\ )_H, \mu_H, [\eta_H])] \in \mathrm{LAD}\,(B, *, V, (\ ,\ ), h)$. Then there is an element $\boldsymbol{\phi} \in \ker^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(E))$ with $(H, (\ ,\ )_H) \cong {}^{\phi}(V^\vee, (\ ,\ )^\vee)$. By lemma 2.1 we may in fact choose $\boldsymbol{\phi}$ to be the image of some

$$\zeta \in \ker(Z^1(\mathrm{Gal}\,(E/\mathbb{Q}), Z(G)(E)) \to H^1(\mathrm{Gal}\,(E/\mathbb{Q}), G(\mathbb{A}_E^\infty)) \oplus H^1(\mathrm{Gal}\,(\mathbb{C}/\mathbb{R}), Z(G)(\mathbb{C}))).$$

Without loss of generality we may assume that $(H, (\ ,\ )_H) = ({}^{\zeta^{-1}}V^\vee, {}^{\zeta^{-1}}(\ ,\ )^\vee)$. Then η_H must arise from $g^\infty \in G(\mathbb{A}_E^\infty)$ with $\zeta^{-1} = {}^{g^\infty}1$. Moreover there is $z_\infty \in Z(G)(\mathbb{C})$ with ${}^{z_\infty}V^\vee \otimes_{\mathbb{Q}} \mathbb{R} = {}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathrm{res}_{\mathbb{C}/\mathbb{R}}\zeta^{-1} = {}^{z_\infty}1$. Then there is $k \in G(\mathbb{R})$ such that $\mu_H = \mathrm{conj}_{z_\infty k} \circ \mu_h = \mathrm{conj}_k \mu_h$, as desired. □

Corollary 5.4. *There is a biholomorphic bijection*

$$S(B, *, V, (\ ,\ ), h)_U(\mathbb{C}) \xrightarrow{\sim} \mathrm{Sh}(G, \psi, [\mu_h^{-1}], U)(\mathbb{C}).$$

Proof: We have exhibited a bijective map, so it is enough to show it is holomorphic, which is a local question. Suppose that $[(A, \lambda, i, [\eta])]$ maps to $\pi_{(1,1)}((\zeta, g), \mu)$. Then for a sufficiently small simply connected neighbourhood Ω of $[(A, \lambda, i, [\eta])]$, $R^1\pi_*\mathbb{Q}$ is constant on Ω and isomorphic to $({}^{\zeta^{-1}}V^\vee, {}^{\zeta^{-1}}(\ ,\ )^\vee)$ and $\eta_x^{\vee,-1}$ is identified with g^∞ for all $x \in \Omega$. The image of $x \in \Omega$ is $({}^{\zeta^{-1}}V^\vee, {}^{\zeta^{-1}}(\ ,\ )^\vee, \mu_x, [g^\infty])$, where $\mu_x \in Y(C)_G$ is characterized by

$$\mathrm{Fil}\,_x^1({}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{C}) = ({}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{C})^{\mu_x(-1)=-1}.$$

As $x \mapsto \mathrm{Fil}\,_x^1({}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{C})$ is a holomorphic map $\Omega \to \mathrm{Gr}(\mathbb{C})$, and $\mu \mapsto ({}^{\zeta^{-1}}V^\vee \otimes_{\mathbb{Q}} \mathbb{C})^{\mu(-1)=-1}$ embeds $Y(C)_G$ biholomorphically as an open subdomain in $\mathrm{Gr}(\mathbb{C})$, we see that $\Omega \to Y(C)_G$ given by $x \mapsto \mu_x$ is holomorphic, as desired.□

It follows from [B2] that this map

$$S(B, *, V, (\ ,\ ), h)_U(\mathbb{C}) \xrightarrow{\sim} \mathrm{Sh}(G, \psi, [\mu_h^{-1}], U)(\mathbb{C})$$

arises from a unique algebraic map

$$S(B, *, V, (\ ,\ ), h)_U \longrightarrow \mathrm{Sh}(G, \psi, [\mu_h^{-1}], U),$$

which is in fact an isomorphism by Zariski's main theorem.

The final assertion of theorem 5.1 follows on unravelling the definitions.

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